Notes on Classifying Spaces of Topological Categories

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Contents

1 Introduction

There is a functorial way to turn a category $\mathscr{C}$ into a topological space $B\mathscr{C}$, called the *classifying space* of $\mathscr{C}$. The classifying space is formed as the geometric realization of the nerve of $\mathscr{C}$, which is a simplicial set that encodes all the objects and morphisms and compositions in $\mathscr{C}$. The idea of the classifying space is that we form a bunch of n-simplices, one for each string of n-composable morphisms, and then glue them together based on how the morphisms compose in $\mathscr{C}$. These constructions are connected to a lot of powerful tools in homotopy theory and algebraic topology more generally; simplicial sets and their realizations show up in things like singular homology, principal G-bundles, model categories, K-theory, and ∞-category theory.

If our category comes with some "extra topological structure," then we want the classifying space construction to keep track of this information somehow. For us, this extra structure will take the form of being *internal* to the category of topological spaces (although Subsection 3.2 discusses how we can relate this to an enriched structure), and we call such categories *topological categories*. To keep track of the topologies on the objects and morphisms, we now need the nerve of $\mathscr{C}$ to be a *simplicial space*, rather than just a simplicial set. With the theory of simplicial spaces in hand, we can explore the theory of classifying spaces of topological categories. In particular, we shall consider the question *What conditions on two topological categories induces a homotopy equivalence on their classifying spaces*? To approach this question, we will explore the notions of Reedy fibrant and Reedy cofibrant simplicial spaces, as well as topological analogues of Quillen's Theorems A and B. We conclude with some interesting examples which show up in algebraic topology, including topological groups and the two-sided bar construction.

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Outline

Section 2 covers the basics of classifying spaces of categories. We begin by briefly reviewing simplicial sets in Subsection 2.1, and then we move on to simplicial spaces in Subsection 2.2. We introduce realization functors in Subsection 2.3 and then define classifying spaces. In the remainder of Section 2, we state and prove Quillen's Theorems A and B and conclude with some common examples including classifying spaces of groups (Subsection 2.5.2), the two-sided bar construction (Subsection 2.5.3), and translation categories (Subsection 2.5.5).

In Section 3, we introduce the idea of a topological category and explore the subtleties of their classifying spaces. We discuss the Reedy model structure on simplicial spaces and how that interacts with nerves of topological categories in Subsection 3.4. Finally, we state and prove versions of Theorems A and B for topological categories (Subsection 3.5) and detail some illustrative examples, including the Čech complex of a map of topological spaces (Example 3.6.3), classifying spaces of topological groups (Example 3.6.1), and framed flow categories (Example 3.6.4).

2 Classifying Spaces

Classifying spaces give us a functorial way to go from the land of categories to the land of spaces, using the machinery of simplicial sets. The outcome of this journey is summarized in the following table:

Cat $\Longrightarrow$	**Top**
Categories	Spaces
Functors	Continuous maps
Natural transformations	Homotopies
Adjunctions	Homotopy equivalences

To understand the classifying space construction, we first need to understand simplicial sets and simplicial spaces (Subsection 2.2) and how to geometrically realize them (Subsection 2.3). The classifying space of a category $\mathscr{C}$ is the realization of a special simplicial set, called the nerve of $\mathscr{C}$, which basically encodes composable morphisms. After introducing classifying spaces, we then state and prove Quillen's Theorem A (Theorem 2.14) and Theorem B (Theorem 2.16). Finally, we get to see this machinery in action through several examples, including the classifying space of a group (Subsection 2.5.2) and the two-sided simplicial bar construction (Subsection 2.5.3).

Our exposition on simplicial sets/spaces follows [Rie08] and [Dug08], and we invite the reader to visit these resources for further discussion of the concepts we present. For classifying spaces specifically, our primary resources are [Wei13, §IV] and [Qui73, §I].

2.1 A refresher on simplicial sets

A simplicial set is a functor $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$, where $\mathbf{\Delta}$ is the simplex category whose objects are finite, non-empty ordinals

$$[n] = \{0, 1, \dots, n\}$$

and whose morphisms are order-preserving maps. As is standard, we write X_n for the set $X[n]$, and call its elements n-simplicies. The simplicial sets form a category, **sSet**, which is just the functor category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$. Specifically, a map $X \to Y$ between simplicial sets is a natural transformation, and it consists of maps $X_n \to Y_n$ that commute with the morphisms of $\mathbf{\Delta}^{\mathrm{op}}$. In fact, it suffices to show that these maps between n-simplices commute with a smaller selection of maps, called the *face* and *degeneracy* maps, which we will introduce presently.

In the category $\mathbf{\Delta}$, for each $n \geq 0$, there are $n+1$ injective *coface* maps $d^i\colon [n-1] \to [n]$, where the superscript indicates which object is not contained in the image. Similarly, there are $n+1$ surjective *codegeneracy* maps $s^j\colon [n+1] \to [n]$, where now the superscript indicates which object in the image is mapped onto twice. Explicitly,

$$d^i(k) = \begin{cases} k & k < i; \\ k+1 & k \geq i, \end{cases} \quad \text{and} \quad s^j(k) = \begin{cases} k & k \leq j; \\ k-1 & k > j, \end{cases}$$

for $0 \leq i, j \leq n$. It is straightforward (albeit a bit tedious) to verify that these morphisms satisfy the following *cosimplicial relations*:

$$d^i d^j = d^j d^{i-1} \qquad i < j, \tag{2.1}$$

$$s^i s^j = s^j s^{i+1} \qquad i \leq j, \tag{2.2}$$

$$s^i d^j = \begin{cases} \text{id} & i = j, j+1, \\ d^j s^{i-1} & i < j, \\ d^{j-1} s^i & i > j+1. \end{cases} \tag{2.3}$$

These maps generate all of the morphisms in $\mathbf{\Delta}$. In fact, any morphism $f\colon [n] \to [m]$ in $\mathbf{\Delta}$ can be expressed uniquely as a composite

$$f = d^{i_k} \dots d^{i_1} s^{j_1} \dots s^{j_{k'}}$$

for $0 \leq i_1 < \cdots < i_k \leq m$ and $0 \leq j_1 < \cdots < j_{k'} \leq n$ such that $n + k - k' = m$. To see this, note that an order-preserving function $f\colon [n] \to [m]$ is determined by its image in $[m]$ and those elements of $[n]$ on which f does not increase. Take $i_1, \dots, i_k \in [m]$ in unique increasing order to be those elements not in the image of f and $j_1, \dots, j_{k'} \in [n]$ (again in unique increasing order) to be the elements on which f does not increase. A quick verification proves the equality in the display above.

Now, the opposite category $\mathbf{\Delta}^{\text{op}}$ has corresponding *face* maps d_i and *degeneracy* maps s_j. If X is a simplicial set, we have

$$d_i := Xd^i\colon X_n \to X_{n-1} \quad \text{and} \quad s_j := Xs^j\colon X_n \to X_{n+1},$$

for $0 \leq i, j \leq n$. Every morphism in $\mathbf{\Delta}^{\text{op}}$ can similarly be expressed as a composition of face and degeneracy maps. These maps satisfy the dual relations to those given above, namely

$$d_j d_i = d_{i-1} d_j \qquad j < i, \tag{2.4}$$

$$s_j s_i = s_{i+1} s_j \qquad j \leq i, \tag{2.5}$$

$$d_j s_i = \begin{cases} \text{id} & i = j, j+1, \\ s_{i-1} d_j & i < j, \\ s_i d_{j-1} & i > j+1. \end{cases} \tag{2.6}$$

These relations are called the *simplicial relations*.

Remark 2.1. To specify a simplicial set, it is enough to provide sets of n-simplices X_n for $n \geq 0$, face maps $d_i \colon X_n \to X_{n-1}$, and degeneracy maps $s_j \colon X_n \to X_{n+1}$ which satisfy the simplicial relations. This gives us a second "definition" of a simplicial set, which is often easier to use in practice.

Example 2.2 (The standard n-simplex)**.** The simplicial set called the *standard n-simplex* is the functor represented by $[n] \in \mathbf{\Delta}$. Letting $y \colon \mathbf{\Delta} \hookrightarrow \mathbf{sSet}$ denote the Yoneda embedding, the standard n-simplex is just the image of $[n]$. That is,

$$\Delta^n := y[n] = \mathbf{\Delta}(-, [n]),$$

so $\Delta^n_k = \mathbf{\Delta}([k], [n])$ by definition. The face and degeneracy maps are given by pre-composition in $\mathbf{\Delta}$ by d^i and s^j, so

$$\begin{array}{cc} d_i \colon \Delta^n_k \to \Delta^n_{k-1} & s_j \colon \Delta^n_k \to \Delta^n_{k+1} \\ ([k] \xrightarrow{f} [n]) \mapsto ([k-1] \xrightarrow{d^i} [k] \xrightarrow{f} [n]) & ([k] \xrightarrow{f} [n]) \mapsto ([k+1] \xrightarrow{s^j} [k] \xrightarrow{f} [n]). \end{array}$$

Non-degenerate k-simplices correspond to the injective maps $[k] \to [n]$ in $\mathbf{\Delta}$; there is a unique non-degenerate n-simplex in Δ^n corresponding to the identity on $[n]$. There are many degenerate simplicies in this data as well: for instance, Δ^0 contains one element in each Δ^0_k, the zero function $[k] \to [0]$, which is degenerate for $k > 0$.

This example Δ^n plays a key role in **sSet**. Since the Yoneda embedding is full and faithful, the maps $f \colon \Delta^n \to \Delta^m$ of simplicial sets are in bijection with the maps $f \colon [n] \to [m]$ in $\mathbf{\Delta}$. The maps $f_k \colon \Delta^n_k \to \Delta^m_k$ are given by post-composition by f. The Yoneda Lemma implies that simplicial maps $\Delta^n \to X$ correspond bijectively to the n-simplices in X, which is to say

$$\mathbf{sSet}(\Delta^n, X) \cong X_n.$$

An n-simplex $x \in X$ can thus be regarded as a map $x \colon \Delta^n \to X$ that sends the unique non-degenerate n-simplex in Δ^n to x. Lower-dimensional simplicies in X can be seen as a composition of maps in Δ^n, post-composed by x.

This perspective helps us actually "visualize" the n-simplices of a simplicial set. Given an n-simplex $x \in X_n$, we can visualize it as an n-dimensional tetrahedron whose $n+1$ vertices are ordered by $0, 1, \ldots, n$ and whose faces are labeled by simplices of the appropriate dimension. The image $d_i(x)$ of x under the i^{th} face map is the $(n-1)$-simplex that does not include the i^{th} vertex of x. Each of the $(n+1)$-simplices $s_0(x), s_1(x), \ldots, s_n(x)$ represent the same simplex geometrically, each with a different degeneracy; the image $s_j(x)$ is the simplex such that collapsing the edge between the j^{th} and $(j-1)^{th}$ vertices to a single point gives the n-simplex x. Accordingly, a simplex is called *degenerate* if it is the image of some s_j, and is *non-degenerate* otherwise. Unlike in a simplicial complex, we allow simplices to be degenerate.

Example 2.3 (Total singular complex)**.** Perhaps unsurpisingly, one standard example of a simplicial set is related to the topological notion of a simplex. Letting $|\Delta^n|$ denote the standard n-simplex[1] in **Top**,

$$|\Delta^n| = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0 \right\} \subseteq \mathbb{R}^{n+1},$$

there is a natural covariant functor $\mathbf{\Delta} \to \mathbf{Top}$ given by $[n] \mapsto |\Delta^n|$. A map $f\colon [n] \to [m]$ induces a map $f_*\colon |\Delta^n| \to |\Delta^m|$ given by $(x_0, \dots, x_n) \mapsto (y_0, \dots, y_n)$ where

$$y_i = \begin{cases} 0 & f^{-1}(i) = \emptyset; \\ \sum_{j \in f^{-1}(i)} x_i & \text{otherwise.} \end{cases}$$

Thus the i^{th} coface map inserts a 0 in the i^{th} coordinate and the j^{th} codegeneracy map adds the x_j and x_{j+1} coordinates. Geometrically, the former inserts $|\Delta^{n-1}|$ as the i^{th} face of $|\Delta^n|$ and the latter projects $|\Delta^{n+1}|$ onto the topological n-simplex orthogonal to its j^{th} face.

Given a topological space Y, the *total singular complex* (or *singular set*) is the simplicial set SY given by $[n] \mapsto \mathbf{Top}(|\Delta^n|, Y)$. Elements of SY_n are the singular n-simplices of Y familiar to algebraic topologists. The face and degeneracy maps are given by pre-composition by d^i and s^j. You can use this functor S to define the singular homology of Y.

Example 2.4. A crucial example of a simplicial set for our purposes is the *nerve* of a category. Given a (small) category $\mathscr{C}$, its *nerve* is the simplicial set $N\mathscr{C}$ whose 0-simplicies the objects of $\mathscr{C}$ and whose n-simplicies are strings of n composable morphisms

$$c_0 \xrightarrow{f_1} c_1 \to \cdots \to c_{i-1} \xrightarrow{f_i} c_i \xrightarrow{f_{i+1}} c_{i+1} \to \dots \xrightarrow{f_n} c_n$$

for $n \geq 1$. The face map $d_i\colon N\mathscr{C}_n \to N\mathscr{C}_{n-1}$ returns the string of $n-1$ composable arrows

$$c_0 \xrightarrow{f_1} c_1 \to \cdots \to c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \to \dots \xrightarrow{f_n} c_n.$$

In the cases that $i = 0, n$, we instead omit that i^{th} arrow. At level $n = 1$, we have only two face maps $d_0, d_1\colon N\mathscr{C}_1 \to N\mathscr{C}_0$ which act like the source and target maps, respectively.

[1]Using the notation $|\Delta^n|$ for the topological simplex might be confusing at first, since this object is more commonly denoted by Δ^n or Δ_n. Our choice will hopefully be justified after we define the geometric realization (Definition 2.9), as we will then see that

$$|\Delta^n| = |\Delta^n|.$$

The expression on the left side of the equality is the geometric realization of the standard n-simplex as a simplicial set, and the expression on the right is the topological n-simplex.

The degeneracy map $s_i \colon N\mathscr{C}_n \to N\mathscr{C}_{n+1}$ returns the string of $n+1$ composable arrows

$$c_0 \xrightarrow{f_1} c_1 \to \cdots \to c_{i-1} \xrightarrow{f_i} c_i \xrightarrow{\mathrm{id}_{c_i}} c_i \xrightarrow{f_{i+1}} c_{i+1} \to \dots \xrightarrow{f_n} c_n.$$

The assignment $\mathscr{C} \to N\mathscr{C}$ defines a functor $N \colon \mathbf{Cat} \to \mathbf{sSet}$, since a functor $F \colon \mathscr{C} \to \mathscr{D}$ induces a natural transformation $NF \colon N\mathscr{C} \to N\mathscr{D}$ by applying F to everything in sight.

Example 2.5. We can regard the objects of $\mathbf{\Delta}$ as categories themselves, more specifically as posets, so $[n]$ is the category with $n+1$ objects and morphisms $i \to j$ whenever $i \leq j$. What is the nerve of $[n]$? After some contemplation, we see that a string of k-composable morphisms in Δ^n is the same thing as a morphism $[k] \to [n]$ in $\mathbf{\Delta}$. That is,

$$(N[n])_k = \mathbf{\Delta}([k], [n])$$

so $N[n]$ is the standard n-simplex Δ^n.

Example 2.6. If G is a group, we can think of it as a category with one object $*$ and a morphism $* \xrightarrow{g} *$ for every $g \in G$. The zeroth level of the nerve is just a point, $NG_0 = *$. For $n \geq 1$, a string of n-composable morphisms is a word of length n in G. Given such a word, $g_1 g_2 \cdots g_n$ the ith face map multiplies $g_i \cdot g_{i+1} = (g_i g_{i+1})$ (or drops g_i if $i = 0, n$) and the jth degeneracy map inserts the identity e at the jth spot.

The definitions and examples we have introduced here barely scratch the surface of the theory of simplicial sets. For example, simplicial sets (Kan complexes, specifically) provide a model for $(\infty, 1)$-categories and so have lots of interesting connections to higher algebra. For more exposition on simplicial sets, we point the reader to [Rie08] or [Fri11]. The textbook [GJ99] also provides an extensive treatment of more advanced simplicial homotopy theory.

2.2 Simplicial spaces

As a generalization of simplicial sets, we can talk about *simplicial objects* in a category $\mathscr{C}$ as functors $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$, which themselves assemble into a category $\mathbf{s}\mathscr{C}$. Our topic of interest is the category **sTop** of *simplicial spaces*, i.e. the category of simplicial objects in **Top**. (Here of course we assume **Top** is some convenient category of spaces, such as compactly generated weak Hausdorff spaces.) A simplicial space is just like a simplicial set, except now there are topologies involved and we ask everything to be continuous.

Definition 2.7. A simplicial space is a functor $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$, which can be specified by the data of

- spaces of n-simplices X_n for $n \geq 0$;

- for each $n \geq 1$, continuous face maps $d_i \colon X_n \to X_{n-1}$ for $i = 0, \ldots, n$;
- for each $n \geq 0$, continuous degeneracy maps $s_j \colon X_n \to X_{n+1}$ for $j = 0, \ldots, n$,

such that the d_i and s_j satisfy the simplicial relations (2.4), (2.5), (2.6). A map of simplicial spaces $X \to Y$ is a continuous natural transformation, that is, a natural transformation whose components are all continuous. Simplicial spaces and maps between them assemble into a category **sTop**.

We can get a simplicial set from a simplicial space by just forgetting the topology, i.e. by post-composing with the forgetful functor $U \colon \mathbf{Top} \to \mathbf{Set}$. Recall that U has both a left *and* a right adjoint, which give a set the discrete and indiscrete topology, respectively. These adjunctions descend to the level of simplicial objects, and thus give us a way to go back and forth between simplicial sets and simplicial spaces.

Many examples of simplicial spaces are interesting because of the information recorded in their geometric realization (Definition 2.9). For this reason, we will postpone most of the examples until after we've defined realization. But we will conclude this subsection with one important example: the simplicial replacement of a diagram.

Example 2.8. We can think of a simplicial space as just a diagram in **Top**, whose indexing category is the small category $\mathbf{\Delta}^{\mathrm{op}}$. But say we have another diagram, $D \colon I \to \mathbf{Top}$, for some other small indexing category I. We can turn D into a simplicial space in a sensible way, using simplicial replacement. Let $srep(D)$ be the simplicial space whose nth level is

$$srep(D)_n = \coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} D(i_n),$$

where $i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n$ is a chain of n morphisms in the indexing category I. So, for example, $srep(D)_0$ is just the coproduct of all the spaces in the image of D in **Top**. The face and degeneracy maps are very similar to those of the nerve of a category. Let $D(i_n)$ be a summand of $srep(D)_n$ corresponding to a chain $i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n$. The jth degeneracy map $srep(D)_n \to srep(D)_{n+1}$ just moves $D(i_n)$ to the identical summand $D(i_n)$ in $srep(D)_{n+1}$ which corresponds to inserting the identity on i_j into the chain. The face maps $srep(D)_n \to srep(D)_{n-1}$ are a bit more complicated. If we look at the jth face map, for $0 \leq j < n$, then we just send $D(i_n)$ to itself $D(i_n)$, but we think of it as living in $srep(D)_{n-1}$ by composing $i_{j-1} \leftarrow i_j \leftarrow i_{j+1}$ to the single arrow $i_{j-1} \leftarrow i_{j+1}$. If $j = n$, then we send $D(i_n)$ to $D(i_{n-1})$ by applying D to the map $i_n \to i_{n-1}$ in I.

As a concrete example of this machinery, if D is just the constant diagram at a point $*$, then $srep(D)$ is the nerve of the indexing category I. Note that in this case $srep(D)$ is a discrete simplicial space, i.e. a simplicial set.

2.3 Realization functors

Some of the most important constructions for simplicial spaces (or sets) are the realization functors, which let us turn simplicial spaces into actual spaces. We can think of forming the geometric realization of X as a simplicial complex where we associate each n-simplex in X_n to a topological n-simplex and glue things together according to the data in X.

Definition 2.9. The *geometric realization* of a simplicial space (or set) X is the colimit

$$|X| = \operatorname{colim}\left(\coprod_{f\colon [n]\to[m]} X_m \times |\Delta^n| \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{[n]} X_n \times |\Delta^n| \right)$$

in **Top**. The map $f_*\colon X_m \times |\Delta^n| \to X_m \times |\Delta^m|$ includes faces by $(x_1, \dots, x_n) \mapsto (y_1, \dots, y_m)$ for

$$y_i = \begin{cases} 0 & f^{-1}(i) = \emptyset \\ \sum_{j \in f^{-1}(i)} x_j & \text{otherwise.} \end{cases}$$

The map $f_*\colon X_m \times |\Delta^n| \to X_n \times |\Delta^n|$ collapses degeneracies via the simplicial structure $Xf\colon X_m \to X_n$.

In practice, it is often useful to use a more concrete description of the geometric realization, which is given by

$$\coprod_{n\geq 0} X_n \times |\Delta^n| / \sim$$

where $(x, s^j y) \sim (s_j x, y)$ and $(x, d^i y) \sim (d_i x, y)$. The d^i inserts a 0 in the i^{th} coordinate and the s^j adds the x_j and x_{j+1} coordinates. Geometrically, the former inserts $|\Delta^{n-1}|$ as the i^{th} face of $|\Delta^n|$ and the latter projects $|\Delta^{n+1}|$ onto the topological n-simplex orthogonal to its j^{th} face. So the first relation ensures that the degeneracies are "glued in" in a compatible way, and the second relation does the same for the faces.

Remark 2.10. The geometric realization of a simplicial *set* is always a CW complex, since $X_n \times \Delta^n$ is just a disjoint union of simplicies, but the geometric realization of a simplicial space may not be. For example, if A is any space which is not a CW complex, then the constant simplicial space $X_n = A$ (where all the face and degeneracy maps are identities) realizes to A itself.

If we only glue in faces, and do not collapse degeneracies, we get a much bigger and more complicated space known as the fat realization.

Definition 2.11. The *fat realization* of X is the colimit

$$||X|| := \operatorname{colim}\left(\coprod_{f\colon [n]\hookrightarrow[m]} X_m \times |\Delta^n| \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{[n]} X_n \times |\Delta^n| \right)$$

in **Top**, where now the coproduct runs over all injections $f\colon [n] \hookrightarrow [m]$.

One of the benefits of working with the fat realization is that it preserves levelwise equivalence. That is, if we have a map of simplicial spaces $X \to Y$ such that each component $X_n \to Y_n$ is a homotopy equivalence, then $||X|| \simeq ||Y||$ (cf. [Dug08, Remark 3.6]). Unfortunately, this is not necessarily true for the regular geometric realization, since we need the additional assumption that X and Y are *Reedy cofibrant* (see Subsection 3.4.1).

We saw that the nerve gives us a way to turn categories into simplicial sets, and now by post-composing with $|-|$, we can turn categories into spaces.

Definition 2.12. The *classifying space* of a category as the realization of its nerve, $B\mathscr{C} := |N\mathscr{C}|$.

We would hope that the functor B might inherit some of the properties of the nerve, such as being fully faithful and commuting with products. Unfortunately, geometric realization makes things a little bit more complicated, since it does not have all the same nice properties as N. The first thing to say is that although B is faithful, it is not full in general (cf. [Qui73, §1]). The second thing is that we can only guarantee that $|-|$ commutes with limits in a convenient category of spaces (e.g. CGWH spaces). The approach we will take, following that of [Seg68] is to work entirely in a convenient category, in which we do indeed have

$$B(\mathscr{C} \times \mathscr{D}) \cong B\mathscr{C} \times B\mathscr{D}.$$

As remarked in [Seg68], the display above holds even in the (inconvenient) category of all spaces if one of $B\mathscr{C}$ or $B\mathscr{D}$ is compact (which holds, for instance, if $\mathscr{C}$ or $\mathscr{D}$ is a finite category). Applying this to $\mathscr{D} = [1]$ and noting that $B[1] \cong I$, we have the following consequence.

Proposition 2.1. *Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be functors with a natural transformation $\eta\colon F \to G$. Then the induced maps $BF, BG\colon B\mathscr{C} \to B\mathscr{D}$ are homotopic via $B\eta$. Thus, equivalent categories have homotopy equivalent classifying spaces.*

In the next section, we will discuss Quillen's Theorems A and B which give other sufficient conditions for a functor to induce an equivalence. But before we do so, we'll make a brief digression into bisimplicial sets since we will need them to prove these theorems.

We have defined simplicial sets and simplicial spaces as functors from $\mathbf{\Delta}^{\mathrm{op}}$ into **Set** or **Top**, respectively, but there's nothing stopping us from replacing these target categories with some other perfectly nice category $\mathscr{C}$. Functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$ are called *simplicial objects* of $\mathscr{C}$ (e.g. simplicial groups or simplicial categories, although we note this latter term may also have other meanings) and the functor category of simplicial objects in $\mathscr{C}$ is denoted $\mathbf{s}\mathscr{C}$. Now, what if $\mathscr{C}$ is already a category of simplicial objects, such as **sSet** or **sTop**? Then the objects of $\mathbf{s}\mathscr{C}$ are functors

$\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathscr{C})$, which (by the tensor-hom adjunction) are the same things as functors $\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$. These objects are called *bisimplicial objects* in $\mathscr{C}$. We will particularly care about *bisimplicial sets*, which are simplicial objects in **sSet**, or equivalently, functors $Z \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$. Note that a bisimplicial set has "built in" simplicial sets if we fix one of the indices:

$$\begin{aligned} Z_{p,*} \colon \Delta^{\mathrm{op}} &\to \mathbf{Top} \text{ for } p \geq 0 \\ [q] &\mapsto Z_{p,q} \end{aligned} \qquad \text{and} \qquad \begin{aligned} Z_{*,q} \colon \Delta^{\mathrm{op}} &\to \mathbf{Top} \text{ for } q \geq 0 \\ [p] &\mapsto Z_{p,q}. \end{aligned}$$

These are called the *left* and *right* simplicial sets of $Z_{*,*}$, respectively. We also have the *diagonal* of Z, which is the simplicial set $diag(Z) \colon [n] \mapsto Z_{p,p}$.

How do we "geometrically realize" a bisimplicial set and get an actual topological space? There are a few different ways we might think to do this, but they turn out to all be homeomorphic. We can't apply the geometric realization functor $|-| \colon \mathbf{sSet} \to \mathbf{Top}$ from Definition 2.9 directly to $Z_{*,*}$, but we can apply it to $Z_{p,*}$ and $Z_{*,q}$ for each $k, n \geq 0$. We can then assemble these spaces into two *new*[2] simplicial spaces: $[p] \mapsto |Z_{p,*}|$ and $[q] \mapsto |Z_{*,q}|$. It turns out that (see [Qui73, §1] for a proof)

$$|[p] \mapsto |Z_{p,*}|| \cong |diag(Z)| \cong |[q] \mapsto |Z_{*,q}||.$$

Intuitively, this says that we can turn $Z_{*,*}$ into a topological space by realizing it twice, and it doesn't matter whether we go left then right, or vice versa; moreover all the information about this space is recorded in the diagonal. We will denote the resulting space by $|Z|$, trusting the reader to know that we don't mean to literally apply $|-|$ to the bisimplicial set $Z_{*,*}$. The most important result for our purposes is the following (see, e.g. [Wei13, Theorem 3.6.1]):

Theorem 2.13. *Let $f_{*,*} \colon Z_{*,*} \to Z'_{*,*}$ be a map of bisimplicial sets. Then*

(i) if each map $f_{p,*} \colon Z_{p,*} \to Z'_{p,*}$ of left simplicial spaces is an equivalence, then so is the induced map $|Z| \to |Z'|$;

(ii) if $Z'_{p,q} = N\mathscr{C}_p$ for some category $\mathscr{C}$ (note Z' is constant in the second factor) and $f^{-1}(c, *) \to f^{-1}(c', *)$ is an equivalence for every $c \to c'$ in $\mathscr{C}$, then each

$$|f^{-1}(c, *)| \to |Z| \to B\mathscr{C}$$

is a homotopy fiber sequence.

2.4 Quillen's Theorems

Theorems A and B were proved by Quillen in [Qui73, §I] as part of his highly influential work on higher algebraic K-theory. Although these two theorems constitute

[2] If anyone has a good name for these, I'd love to know.

just a tiny portion of Quillen's amazing contribution to algebraic topology, they have proven to be extremely useful in studying the homotopy groups of categories.

By functoriality of $B\colon \mathbf{Cat} \to \mathbf{Top}$, we can determine when certain functors will realize to equivalences. For instance, any adjoint functor realizes to an equivalence, with the relevant homotopies coming from the co/unit of the adjunction. More generally, any functor which admits *lax inverse*[3] will turn into an equivalence under B. Quillen's Theorem A gives another sufficient condition for a functor to realize to an equivalence, namely if all the "fibers" are contractible. Theorem B says that even if the fibers are not contractible, we may still be able to use them to model the homotopy fiber.

2.4.1 Theorem A

Theorem A will detect whether a functor $F\colon \mathscr{C} \to \mathscr{D}$ is a homotopy equivalence (meaning BF is) by looking at the classifying spaces of the comma categories $d{\downarrow}F$ for $d \in \operatorname{Ob}\mathscr{D}$. The objects of $d\downarrow F$ are pairs $(c, g\colon d \to Fc) \in \operatorname{Ob}\mathscr{C} \times \operatorname{Hom}\mathscr{D}$. A morphism between (c, g) and (c', g') in $d\downarrow F$ is a map $f\colon c \to c'$ such that the following triangle commutes in $\mathscr{D}$:

$$\begin{array}{ccc} & d & \\ {\scriptstyle g}\swarrow & & \searrow{\scriptstyle g'} \\ Fc & \xrightarrow[Ff]{} & Fc' \end{array}.$$

For example, if F is the identity on $\mathscr{C}$, then $d{\downarrow}F$ is just the undercategory of $d \in \mathscr{C}_0$.

Theorem 2.14 (Quillen's Theorem A)**.** *If $d{\downarrow}F$ is contractible for every $d \in \operatorname{Ob}\mathscr{D}$, then F induces an equivalence $BF\colon B\mathscr{C} \to B\mathscr{D}$.*

Remark 2.15. There is also a dual statement of this theorem using $F{\downarrow}d$ instead.

Proof of Theorem A. To prove Theorem A, we use the bisimplicial set $Z_{*,*}$ whose (p, q)-bisimplices are diagrams

$$(d_p \to \cdots \to d_0 \to F(c_0), c_0 \to \cdots \to c_q).$$

The i^{th} face map in the p-direction (or q-direction) omits d_i (or c_i) by composing the two relevant arrows. The utility of this bisimplicial set is actually pretty amazing, as we shall soon see.

[3]Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor. We say $G\colon \mathscr{D} \to \mathscr{C}$ is a *lax inverse* for F if there are natural transformations connecting GF to $\mathrm{id}_{\mathscr{C}}$ and FG to $\mathrm{id}_{\mathscr{D}}$. Unlike adjunctions, there is no requirement on the direction of the natural transformations and they are not required to satisfy any further identities.

As a first question, we can investigate the realization of this bisimplicial set. From what we said earlier, it suffices to think about the geometric realization of the diagonal,

$$Z_{p,p} = (d_p \to \cdots \to d_0 \to F(c_0), c_0 \to \cdots \to c_p).$$

The clever observation is that this diagonal is actually the nerve of a category, which Quillen denotes $\mathscr{S}(f)$. This objects of this category are triples $(d, f\colon d \to F(c), c)$ where $d \in \operatorname{Ob}\mathscr{D}$, $c \in \operatorname{Ob}\mathscr{C}$, and $f \in \mathscr{D}(d, F(c))$. A morphism from (d, f, c) to (d', f', c') is a pair of morphisms $u\colon d' \to d$ and $v\colon c \to c'$ which make the following square commute:

$$\begin{array}{ccc} d & \xleftarrow{\;u\;} & d' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ F(c) & \xrightarrow[Fv]{} & F(c') \end{array}$$

For example, if F is the identity then $\mathscr{S}(F)$ is the twisted arrow category. A p-simplex in $N\mathscr{S}(F)$ is a commutative diagram of p squares:

$$\begin{array}{ccccccc} d_0 & \xleftarrow{\;u_1\;} & d_1 & \xleftarrow{\;u_2\;} & \dots & \xleftarrow{\;u_p\;} & d_p \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_p} \\ F(c_0) & \xrightarrow[Fv_1]{} & F(c_1) & \xrightarrow[Fv_2]{} & \dots & \xrightarrow[Fv_p]{} & F(c_p) \end{array}$$

but since this diagram is commutative, we can forget the data of all the f_i for $i \neq 0$ and define them instead using composition. That is, the data of the diagram is exactly equivalent to a tuple $(d_p \xrightarrow{u_p} \dots \xrightarrow{u_1} d_0 \xrightarrow{f_0} F(c_0), c_0 \xrightarrow{v_1} \dots \xrightarrow{v_p} F(c_p))$. And the face and degeneracy maps do the right thing, but that is pretty straightforward to check once you're comfortable with all the machinery.

The idea is to map $Z_{*,*}$ to both $N\mathscr{C}$ and $N\mathscr{D}$ and show these maps realize to equivalences using Theorem 2.13. If we project in the q-direction, forgetting the diagram $d_p \to \cdots \to d_0 \to F(c_0)$, this gives us a map $Z_{p,q} \to N\mathscr{C}_q$, which we can assemble into a map of bisimplicial sets (where we think of $N\mathscr{C}$ as a bisimplicial set which is constant in the first factor). This induces a map $B\mathscr{S}(F) \cong |Z| \to B\mathscr{C}$. We'd like to show this map is actually a homotopy equivalence, which we will do with Theorem 2.13(ii).[4] The fiber of $Z_{p,q} \to N\mathscr{C}_q$ over $c \in \mathscr{C}$ is just the collection of sequences $d_p \to \cdots \to d_0 \to F(c)$ in $\mathscr{D}$, i.e. the p^{th} level of $N(\mathscr{D} \downarrow F(c))$. Since the category $D \downarrow F(c)$ has an final object, its classifying space is contractible, and so we can apply Theorem 2.13(ii) to see that $|Z| \to B\mathscr{C}$ is in fact an equivalence.

Now we need to connect $|Z|$ to $B\mathscr{D}$, and the argument is very similar, except now we project from $Z_{p,q} \to N(\mathscr{D}^{\text{op}})_p$ by sending $(d_0 \to \cdots \to d_p \to F(c_0), c_0 \to \cdots \to c_q)$ to just $d_0 \to \cdots \to d_p$. This induces a map $B\mathscr{S}(F) \cong |Z| \to B(\mathscr{D}^{\text{op}}) \cong B\mathscr{D}$, which we would like to show is an equivalence. Now we will use Theorem 2.13(ii)

[4] The theorem is stated for $Z'_{p,q} = N\mathscr{C}_p$ but we can equally well assume $Z'_{p,q} = N\mathscr{C}_q$.

and our assumption that all the $d{\downarrow}F$ are contractible. In particular, the fiber over d is the the nerve of the category $d \downarrow F$, which we've assumed to be contractible. Thus, applying Theorem 2.13(ii), we can conclude the map $B\mathscr{S}(F) \to B\mathscr{D}$ is also an equivalence.

We've shown this works for arbitrary F, and in particular it works for $F = \mathrm{id}_{\mathscr{D}}$, which gives us the commutative diagram

$$\begin{array}{ccccc} \mathscr{C} & \xleftarrow{\sim} & \mathscr{S}(F) & \xrightarrow{\sim} & \mathscr{D}^{\mathrm{op}} \\ {\scriptstyle F}\downarrow & & \downarrow & & \| \\ \mathscr{D} & \xleftarrow[\sim]{} & \mathscr{S}(\mathrm{id}_{\mathscr{D}}) & \xrightarrow[\sim]{} & \mathscr{D}^{\mathrm{op}} \end{array}$$

where the functor $\mathscr{S}(F) \to \mathscr{S}(\mathrm{id}_{\mathscr{D}})$ sends a triple $(d, f\colon d \to F(c), c)$ to $(d, f, F(c))$. Thus F induces an equivalence on classifying spaces, as desired. □

2.4.2 Theorem B

Theorem B gives us a sufficient condition to describe the homotopy fiber of $BF\colon B\mathscr{C} \to B\mathscr{D}$ as a classifying space, which then lets us study BF in terms of homotopy groups. To state Theorem B, we need to recall what homotopy pullback squares are. Recall that a pullback square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & Z \end{array}$$

of spaces is a *homotopy pullback* (or *homotopy Cartesian*) if W is a model for the homotopy limit $X \times^h_Z Y$, meaning there is an equivalence from W to

$$X \times^h_Z Y = \{(x, y, \alpha) \in X \times Y \times Z^I \mid g(x) = \alpha(0), f(y) = \alpha(1)\}$$

(or some other model for the homotopy pullback). This holds, for instance, if either of the maps out of W are fibrations. We say commutative square of categories is a homotopy pullback if the corresponding square of their classifying spaces is.

For any choice of basepoint in Z, the homotopy pullback $X \times^h_Z Y$ sits naturally in a fibration sequence

$$\Omega Z \to X \times^h_Z Y \to X \times Y.$$

By the long exact sequence of a fibration (and remembering $\pi_k(\Omega Z) = \pi_{k+1}(Z)$), this gives us a Mayer-Vietoris sequence of homotopy groups,

$$\cdots \to \pi_{k+1}(Z) \to \pi_k(X \times^h_Z Y) \to \pi_k(X) \oplus \pi_k(Y) \to \pi_k(Z) \to \ldots$$

which tells us that taking homotopy pullbacks preserves pointwise-weakly equivalent diagrams.

Theorem 2.16 (Theorem B)**.** *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor and suppose that for every morphism $d \to d'$ in $\mathscr{D}$, the induced functor $d'\downarrow F \to d\downarrow F$ is a homotopy equivalence. Then the pullback square*

$$\begin{array}{ccc} d\downarrow F & \longrightarrow & \mathscr{C} \\ {\scriptstyle F'}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ d\downarrow \mathscr{D} & \longrightarrow & \mathscr{D} \end{array}$$

is a homotopy pullback.

Here, the horizontal arrows are given by projection onto the first coordinate and the vertical arrow F' sends $(c, g\colon d \to Fc)$ to just $g\colon d \to Fc$. Thus for any choice of $d \in \mathscr{D}$ and $c \in F^{-1}(d)$ in the fiber, we can apply the Mayer-Vietoris sequence and use contractibility of $B(d\downarrow D)$ to get

$$\cdots \to \pi_{k+1}(\mathscr{D}, d) \to \pi_k(d\downarrow F, (c, \mathrm{id}_d)) \to \pi_k(\mathscr{C}, c) \to \pi_k(\mathscr{D}, d) \to \dots.$$

Of course, by the homotopy group of a category we really mean the homotopy group of its classifying space. The display above says that $B(d\downarrow F)$ is actually a model for the homotopy fiber of F.

The proof of Theorem B uses a lot of the same ideas as the proof of Theorem A. In particular, we use the same bisimplicial space $Z_{*,*}$ and again have that the projection $Z_{p,q} \to N\mathscr{C}_q$ is an equivalence. We can also consider the projection $Z_{p,q} \to N(\mathscr{D}^{\mathrm{op}})_p$ and apply Theorem 1.24(ii) (using our assumption that the induced functors $d'\downarrow F \to d\downarrow F$ are equivalences), which gives us a homotopy fiber sequence

$$B(d\downarrow F) \to B\mathscr{S}(F) \to B\mathscr{D}^{\mathrm{op}}.$$

For Theorem A, we assumed $d\downarrow F$ was contractible, but we may not have that here. However, this does tell us that

$$\begin{array}{ccc} d\downarrow F & \longrightarrow & \mathscr{S}(F) \\ \big\downarrow & & \big\downarrow \\ \{d\} & \longrightarrow & \mathscr{D}^{\mathrm{op}} \end{array}$$

is a homotopy pullback square. This fits into a larger diagram

$$\begin{array}{ccccc} d\downarrow F & \longrightarrow & \mathscr{S}(F) & \xrightarrow{\sim} & \mathscr{C} \\ \big\downarrow & \mathrm{I} & \big\downarrow & \mathrm{II} & \big\downarrow{\scriptstyle F} \\ d\downarrow \mathscr{D} & \longrightarrow & \mathscr{S}(\mathrm{id}_{\mathscr{D}}) & \xrightarrow[\sim]{} & \mathscr{D} \\ {\scriptstyle\sim}\big\downarrow & \mathrm{III} & \big\downarrow{\scriptstyle\sim} & & \\ \{d\} & \longrightarrow & \mathscr{D}^{\mathrm{op}} & & \end{array}$$

as square I+III. Note that I+II is exactly the diagram we care about for Theorem B. Now we use the special 2-out-of-3 property for homotopy pullbacks: Because I+III and III are homotopy pullbacks, I must also be. Then because I and II are homotopy pullbacks, so is I+II, as desired.

As with Theorem A, there is a dual version of Theorem B for $F\downarrow d$. In [Qui73, §1] (or see [Wei13, §IV.3.7.3ish]), Quillen discusses some variations of Theorems A and B using co/fibered functors $F\colon \mathscr{C} \to \mathscr{D}$ (which are in one-to-one correspondence with covariant/contravariant functors $\mathscr{D} \to \mathbf{Cat}$).

Remark 2.17. In [WD89, §6], Dwyer-Kan-Smith generalize Quillen Theorem B to "Theorems B$_n$" where the category $F\downarrow d$ is replaced with a category $F\downarrow_n d$ whose objects are pairs (c, D) where D is a zig-zag of length n connecting Fc and d,

$$D = Fc = d_n \cdots \to d_2 \leftarrow d_1 \to d.$$

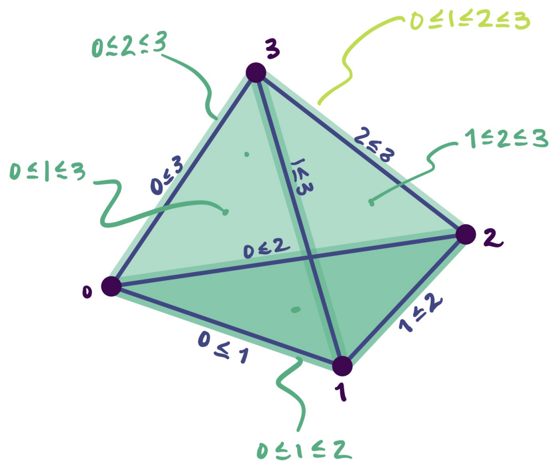
Their theorem [WD89, Theorem 6.2] states that if a functor $F\colon \mathscr{C} \to \mathscr{D}$ has property B_n (every map $d \to d'$ induces an equivalence $F\downarrow_n d \to F\downarrow_n d'$) then $F\downarrow_n d$ models the homotopy fiber of f.

2.5 Some Examples

In this section, we will work through several examples (many taken from [Wei13, §IV.3]). These are presented in no particular order and in general should be able to be read independently of one another.

2.5.1 Standard n-simplex

Let us return to the poset category $[n]$, whose objects are the set $0, 1, \ldots, n$. Recall from Example 2.5 that $N[n] = \Delta^n$. So we know $B[n] = |\Delta^n|$, where the right side is the realization of the simplicial set Δ^n. As the notation suggests, we expect this space to be the topological n-simplex. To get an idea of how this works, first note that the *non-degenerate* elements of $N[n]_k$ are in bijection with size-k subsets of $[n]$. Using the concrete description, we see that we get one topological k-simplex for each collection of k elements, and the relation $\sim$ tells us to glue the faces together in precisely the right way. For example, when $n = 3$:

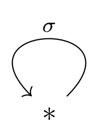


2.5.2 Classifying space of a group

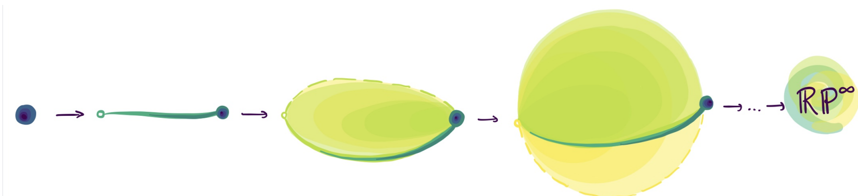
Let G be a well-based topological group, i.e. a topological group such that the inclusion of the identity is a cofibration. We can think of G as a category with one object $*$ and a morphism $* \xrightarrow{g} *$ for every $g \in G$; this is an example of an internal topological category (since the source, target, and identity maps are trivially continuous and composition is continuous by assumption). The zeroth level of the nerve is just a point, $NG_0 = *$. For $n \geq 1$, a string of n-composable morphisms is a word of length n in G, i.e. $NG_n = G^n$. Given such a word, $g_1 g_2 \cdots g_n$ the ith face map multiplies $g_i \cdot g_{i+1} = (g_i g_{i+1})$ (or drops g_i if $i = 0, n$) and the jth degeneracy map inserts the identity e at the jth spot.

The resulting classifying space BG is important for a number of reasons, one of which is closely related to bundle theory. In particular, this classifying space machinery provides a (functorial!) model for the base space of the universal principal G-bundle $EG \to BG$. That is, every principal G-bundle over X arises as a pullback of some map $f \colon X \to BG$. From bundle theory, we know some specific examples of BG; for instance, if G is a discrete group, then BG is an Eilenberg-Mac Lane space $K(G, 1)$ (this follows from the long exact sequence for a fibration). Note that if G is discrete, then NG is a simplicial *set* and so BG is actually a CW complex (cf. Remark 2.10).

Let's see a sub-example of this with the simplest non-trivial group possible, $G = \mathbb{Z}/2$. From bundle theory, we expect $BG = \mathbb{RP}^\infty$ — this space is already quite complicated, even for such a small group! As a category, $\mathbb{Z}/2$ looks like



where σ is the non-identity element of $\mathbb{Z}/2$. This implies that the nerve of $\mathbb{Z}/2$ only has one non-degenerate simplex at each level n, corresponding to the n-fold composition of σ. Thus the classifying space $B\mathbb{Z}/2$ has one n-simplex in it for every $n \geq 0$. The conscientious reader can check that the gluing relations in the geometric realization (along with the fact that $\sigma^2 = \mathrm{id}$) gives $B\mathbb{Z}/2$ the same CW description as $\mathbb{RP}^\infty$, with one cell in each dimension $n \geq 0$.



2.5.3 Two-sided bar construction

We can also model the nerve of a category using the incredibly (perhaps even unreasonably) helpful two-sided simplicial bar construction.

Definition 2.18. Let I be our (small) indexing diagram (discrete for now) and let $F\colon I \to \mathbf{Top}$ and $G\colon I^{\mathrm{op}} \to \mathbf{Top}$ be two functors, i.e. I-shaped diagrams in $\mathbf{Top}$. The *two-sided simplicial bar construction* is the simplicial space with n^{th} level

$$B_n(F, I, G) := \coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} F(i_0) \times G(i_n).$$

Most of the face and degeneracy maps come from the nerve NI, with the exception of d_0 and d_n. Specifically, $s_j\colon B_n(F,I,G) \to B_{n+1}(F,I,G)$ just inserts the identity $i_j \to i_j$ and $d_j\colon B_n(F,I,G) \to B_{n-1}(F,I,G)$ composes $i_{j-1} \to i_j \to i_{j+1}$ for $0 < j < n$ (so the values of F and G are unchanged in these cases); d_0 maps $F(i_0)$ to $F(i_1)$ via the map $F(i_0 \to i_1)$ and similarly d_n maps $G(i_n)$ to $G(i_{n-1})$ via the map $G(i_{n-1} \to i_n)$. The *two-sided bar construction* $B(F,I,G)$ is the realization of this simplicial space.

Note that if $F = * = G$ are trivial functors then $B(*, I, *) = BI$ is just the classifying space of I. We can think of $B(F,I,G)$ like BI "weighted" or "fattened up" by the spaces picked out by F and G. A crucial property of the two-sided simplicial bar construction is that if F and G are both pointwise cofibrant, then $B_*(F, \mathscr{C}, G)$ is Reedy cofibrant (cf. [Dug08, Proposition 11.6]). We will discuss Reedy cofibrancy in more detail in Subsection 3.4.1, but the important consequence in this context is the following:

Proposition 2.2. *Suppose there are weak natural equivalences $F \to F'$ and $G \to G'$ between pointwise cofibrant diagrams. Then the induced map $B(F,I,G) \to B(F',I,G')$ is a weak equivalence.*

In the case that $F = *$ is the trivial diagram, then the two-sided bar simplicial construction turns the I-shaped diagram G into the simplicial replacement we saw in Example 2.8. One remarkable property of the simplicial replacement of G is that it is a Reedy cofibrant simplicial space, regardless of whether G is pointwise cofibrant (cf. [Dug08, Remark 4.9]). By Theorem 3.14, this means the homotopy type of the the realization does not change when we replace G with a pointwise equivalent diagram. This ties in nicely with the fact that the realization of the simplicial replacement of G is a model for its homotopy colimit,

$$B(*, I, G) \simeq \operatorname{hocolim} G.$$

The Reedy cofibrancy of simplicial replacement also encodes the fact that the homotopy colimit preserves pointwise weak equivalences between diagrams. We will return to Reedy cofibrancy, simplicial replacement, and homotopy colimits for a specific example in Example 3.16.

But we can actually use the two-sided bar construction to also model the total space EG of the universal principal G-bundle. Recall that EG is characterized by the property of being a contractible free G-space, and then BG is the orbit space EG/G. To use the two-sided bar construction, we think of G as a functor (more appropriately, a left G-module) $G \to \mathbf{Top}$ which sends the unique object $*$ to the space G (forgetting the group structure) and every morphism $g \in G$ gets sent to multiplication on the left by g. It turns out that G acts freely on the right of $B(*, G, G)$, and when we quotient out by this action we get precisely $B(*, G, *) = BG$. Showing $B(*, G, G)$ is contractible takes a bit more work, but is proved e.g. in [May75, §7]. This means that $B(*, G, G) \to B(*, G, *)$ is a (the) universal principal G-bundle where the projection map is induced by $\pi_G \colon G \to *$; we will see these ideas return at the end of the next example.

2.5.4 Čech complex of a map

Let $f \colon X \to Y$ be a map of topological spaces. The *Čech complex* of f is the simplicial space $C(f) \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$ whose space of n-simplicies is the n-iterated pullback

$$C(f)_n = X \times_Y \cdots \times_Y X$$

with $n+1$ factors. Given an element $(x_0, \ldots, x_n) \in C(f)_n$, the ith face map omits x_i and the jth degeneracy map repeats x_j. Note that if $(x_0, \ldots, x_n)$ is an element of $C(f)_n$, then each x_i has the same image in Y under f, so this induces a map $|C(f)| \to Y$. It turns out that if f has a section, then this map is a homotopy equivalence. (One way to prove this statement is to use *augmented simplicial sets* and *source/sink-like contracting homotopies*. See [Dug08, §3.10] for more details.)

If X and Y are discrete, we can actually view $C(f)$ as the nerve of a certain category. In the non-discrete case, this will be the nerve of a *topological* category, which we will discuss further in Section 3. Let $\mathscr{C}(f)$ be the category with one object for each element of X and a unique morphism $x_0 \to x_1$ if and only if $f(x_0) = f(x_1)$. The nth level of the nerve is exactly $C(f)_n$ since

$$(X \times_Y X) \times_X (X \times_Y X) \cong X \times_Y X \times_Y X,$$

and so $B\mathscr{C}(f) = |C(f)|$.

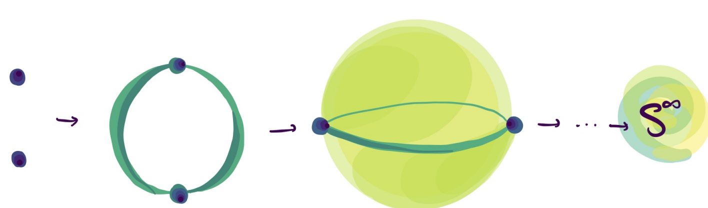
We can also model $EG \to BG$ using this Čech complex machinery. One useful thing that this approach does for us, as opposed to the two-sided bar construction, is it builds EG as the classifying space of a category. This is also taken up in [Seg68, §3], although we note he does not discuss this in the language of Čech complexes.

Let $\pi_G \colon G \to *$ be the projection map, and think of G as a G-space via the group operation. Then if we form the Čech complex of π_G this also has a G action (the diagonal one). Alternatively, we can think of the Čech complex as the nerve of the category $\mathscr{C}(\pi_G)$, with objects G and morphisms $G \times G$. This category is sometimes denoted $\hat{G}$ or $\overline{G}$ and called the *chaotic category* associated to the group G. The realization $|C(\pi_G)| = B\mathscr{C}(\pi_G)$ turns out to be a contractible (because π_G has a section) free (because G acts freely on itself) G-space and hence models EG. It is worth noting that nothing really new is going on here: in this case we can interpret $C(\pi_G)_n$ as just $B_n(*, G, G)$.

For example, when $G = \mathbb{Z}/2$, we can build $E\mathbb{Z}/2$ using the Čech complex of $\pi_{\mathbb{Z}/2} \colon \mathbb{Z}/2 \to *$. Since this map has a section, we know $E\mathbb{Z}/2$ will be equivalent to the base space $*$, i.e. contractible. The category $\mathscr{C}(\pi_{\mathbb{Z}/2})$ looks like

$$* \underset{g}{\overset{f}{\rightleftarrows}} *$$

since both points have the same image under the projection. The nerve of this category has two non-degenerate simplices at each level, corresponding to alternating f and g (either starting with f or starting with g). Since fg and gf are both the identity, the gluing relations in the geometric realization imply that $B\mathscr{C}(\pi_{\mathbb{Z}/2}) = E\mathbb{Z}/2$ has the same CW description as S^∞, with two cells in each dimension.



2.5.5 Translation categories

If X is a G-set for some group G, we define its *translation category* $\underline{X}$ to have object set X and homsets $\underline{X}(x, x') = \{g \in G \mid g \cdot x = x'\}$. This category is built to encode the action of G on X, as demonstrated by the following proposition.

Proposition 2.3. *The classifying space $B(\underline{X})$ is homotopy equivalent to $\coprod BG_x$ where the coproduct is taken over representatives $[x] \in X/G$.*

To prove this, we first observe that a G-set X decomposes as a disjoint union of its orbits, each of which can be written as G/G_x for some x in the orbit. Thus after choosing coset representatives (which doesn't affect the homotopy type, since if $[x] = [y]$ then G_x is conjugate to G_y), we can write $X = \coprod_{[x] \in X/G} G/G_x$. This identification carries over to the translation categories, so it suffices to prove that $B(\underline{G/H}) \simeq BH$ for $H \leq G$, where $\underline{G/H}$ is the translation category for the transitive action of G on G/H and H is the usual one-object category. Consider the inclusion $i\colon H \to \underline{G/H}$ where the single object maps to the identity coset eH and each morphism $h \in H$ maps to itself. We claim this functor defines an equivalence of categories. Unfortunately its tricky to give a well-defined functor in the other direction, so instead we'll show that this inclusion is essentially surjective and fully faithful. For the first point, we simply note that $eH \cong gH$ for each coset (via multiplication by g or g^{-1}). For fully faithful, we note that $H(*, *) \cong H$ and $\underline{G/H}(eH, eH) \cong H$ naturally, so we're done.

Remark 2.19. We can define π_0 of a category to be the set of objects modulo the equivalence relation generated by the morphisms. So if we look at $\underline{X}$, we see that $\pi_0(\underline{X}) = X/G$. More generally, we can define translation categories for functors $X\colon I \to \mathbf{Set}$ where I is any small category. The objects are pairs (i, x) for $i \in \operatorname{Ob} I$ and $x \in X(i)$, and a morphism $(i, x) \to (i', x')$ is a map $f\colon i \to i'$ such that Xf maps x to x'. By the description of π_0, we see that $\pi_0(\underline{X}) = \operatorname{colim}_I X$.

As another example, we could consider G as an H-set for $H \leq G$, again acting by left multiplication. Now our object set is G and the morphisms from g_1 to g_2 are the $h \in H$ such that $hg_1 = g_2$, so there is at most one morphism (depending on whether $g_2 g_1^{-1} \in H$ or not). Now the proposition tells us that $B(\underline{G})$ is equivalent to a coproduct over $[g] \in G/H$ of stabilizers. But every $g \in G$ has trivial stabilizer in H, hence $B(\underline{G}) \simeq G/H$.

If we write $i\colon H \to G$ for the inclusion of the associated categories, then we can see that $* {\downarrow} i = \underline{G}$:

$$\operatorname{Ob}(i{\downarrow}*) = \{(*, g\colon * \to *) \in H \times \operatorname{Hom} G\} \cong G,$$
$$\operatorname{Hom}(i{\downarrow}*) = \{h \in H \mid hg_1 = g_2\} = \{h \in H \mid h = g_2 g_1^{-1}\}.$$

Note that for any $g \in G$, the induced automorphism of $*{\downarrow}i$ given by pre-composing multiplication-by-g (aka multiplying on the right) is an equivalence (in fact, an isomorphism). Thus we can apply Theorem B to conclude that the homotopy fiber of $BH \to BG$ is $B(* \downarrow i) = B(\underline{G}) \simeq G/H$. Note that the fiber of i, which by definition is the subcategory of H which gets mapped to $* \xrightarrow{e} *$, is just the trivial subcategory and hence contractible. The homotopy fiber is somehow giving us the "right" information here.

A specific example that falls under both of these umbrellas is G acting on itself, where the objects are G and there is exactly one morphism $g_1 \to g_2$ (the element

$g_2 g_1^{-1}$). Using either two methods outlined above (thinking about $B(\underline{G/e})$ or $B(* \downarrow \mathrm{id}_G)$), we see that $B(\underline{G}) \simeq B(*) = *$. Since G acts freely on itself, $B(\underline{G})$ is also a free G-space and hence a model for the universal cover EG. This gives us yet another way to build EG, but once again this is actually the same model we've seen before: the category $\underline{G}$ is the same as the category $\mathscr{C}(\pi_G) = \overline{G}$ we saw in the previous example.

2.5.6 Subdivision categories

The subdivision category of $\mathscr{C}$ is like "$\mathscr{C}$ shifted by a degree," where now the objects are morphisms and the morphisms are "morphisms between morphisms" in the form of commutative squares. The author personally prefers the name *twisted arrow category* because it lines up more closely with *arrow category*.

Definition 2.20. Given a small category $\mathscr{C}$, the *twisted arrow category* (or *subdivision category*) of $\mathscr{C}$, denoted $\mathrm{tw}(\mathscr{C})$, the category whose objects are morphisms of $\mathscr{C}$, written vertically, and whose morphisms are commutative squares. That is, a morphism from $a \to b$ to $c \to d$ is given by

$$\begin{array}{ccc} a & \longleftarrow & c \\ \downarrow & & \downarrow \\ b & \longrightarrow & d \end{array}$$

For example, the twisted arrow category of the poset category $[n]$ has objects (i, j) for every $0 \leq i \leq j \leq n$, and a morphism $(i \to j) \leftrightarrows (i' \to j')$ whenever $i \geq i'$ and $j \leq j'$.

Remarkably, twisting the category in this way does not affect the classifying space, as $B\mathscr{C}$ and $B\mathrm{tw}(\mathscr{C})$ are actually homeomorphic. To see this, we use an operation on simplicial spaces known as *Segal's edgewise subdivision.*

The category Δ has a (non-symmetric) monoidal structure via the join $\star$. Given linearly ordered sets I, J, their *join* $I \star J$ is the set $I \amalg J$ with the original orderings on I and J, along with the additional condition that $i < j$ for all $i \in I$, $j \in J$. For example, we can think of $[n] \star [m]$ as

$$\overline{0} < \overline{1} < \cdots < \overline{n} < 0 < 1 < \cdots < m,$$

where the overline is merely meant to distinguish between the elements of $[n]$ and those of $[m]$. Now, let $\varepsilon\colon \mathbf{\Delta} \to \mathbf{\Delta}$ be given by $\mathrm{op} \star \mathrm{id}$, so $\varepsilon([n]) = [n]^{\mathrm{op}} \star [n] \cong [2n+1]$, where $[n]^{\mathrm{op}}$ is meant to indicate $[n]$ with reversed ordering. Hence we can think of $\varepsilon([n])$ as

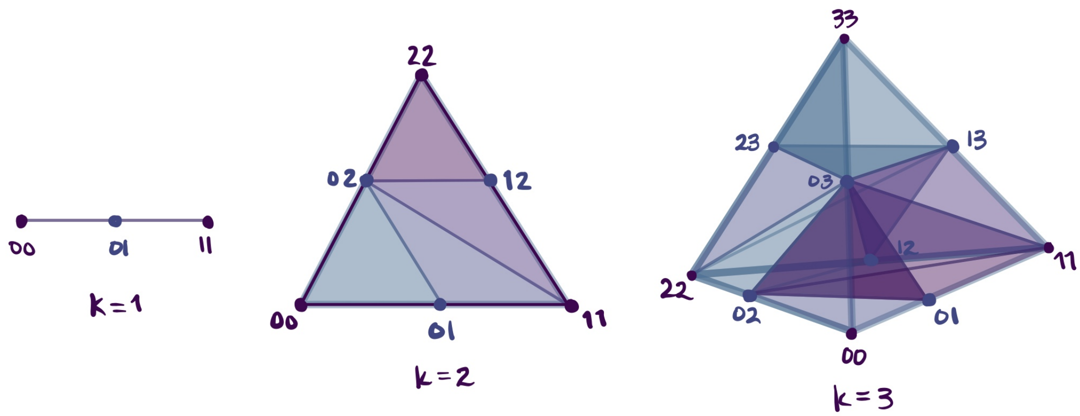
$$\overline{n} < \overline{n-1} < \cdots < \overline{1} < \overline{0} < 0 < 1 < \cdots < n-1 < n.$$

Definition 2.21. Given a simplicial set X, the *edgewise subdivision* of X is the simplicial set $\mathrm{sd}(X) = X \circ \varepsilon$, with each component $\mathrm{sd}(X)_n \cong X_{2n+1}$. The vertices of $\mathrm{sd}(X)$ are the edges of X, and an edge of $\mathrm{sd}(X)$ from $a \to b$ to $c \to d$ can be viewed as a commutative diagram,

$$\begin{array}{ccc} a & \longleftarrow & c \\ \big\downarrow & & \big\downarrow \\ b & \longrightarrow & d \end{array}.$$

The edgewise subdivision is thus a functor $\mathrm{sd}\colon \mathbf{sSet} \to \mathbf{sSet}$ specified by $\mathrm{sd}(X)_n = X_{2n+1}$ and the structure maps $\mathrm{sd}(d_i) = d_{n-i} \circ d_{n+1+i}\colon \mathrm{sd}(X)_n \to \mathrm{sd}(X)_{n-1}$ and $\mathrm{sd}(s_j) = s_{n-j} \circ s_{n+1+j}\colon \mathrm{sd}(X)_n \to \mathrm{sd}(X)_{n+1}$.

For the case $X = \Delta^k$, the edgewise subdivision divides Δ^k into 2^k non-degenerate k-simplicies. We can visualize this as literally subdividing the topological simplicies in the realization, as is illustrated below for $k = 1, 2, 3$.



A collection of vertices $\{v_{i_0,j_0}, \dots, v_{i_{k+1},j_{k+1}}\}$ determines a simplex when $i_0 \geq \cdots \geq i_{k+1}$ and $j_0 \leq \cdots \leq j_{k+1}$ (and each of the numbers $0, \dots, k$ appears at least once).

In the case when $X = N\mathscr{C}$ is the nerve of some category, the definition of the edgewise subdivision strongly resembles the twisted arrow construction. The two notions are linked together by the following observation:

Proposition 2.4. *If $\mathscr{C}$ is a small category, then* $\mathrm{sd}(N\mathscr{C}) \cong N\,\mathrm{tw}(\mathscr{C})$.

Proof. We see that $\mathrm{sd}(N\mathscr{C})_0 \cong N\mathscr{C}_1 = \mathrm{Hom}\,\mathscr{C} = \mathrm{Ob}\,\mathrm{tw}(\mathscr{C}) = N\,\mathrm{tw}(\mathscr{C})_0$. Similarly,

$$\mathrm{sd}(N\mathscr{C})_1 \cong N\mathscr{C}_3 = \{\cdot \to \cdot \to \cdot \to \cdot\} = \left\{ \begin{array}{ccc} \cdot & \longleftarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \cdot \end{array} \right\} = N\,\mathrm{tw}(\mathscr{C})_1$$

and so on. It suffices to show that $N\,\mathrm{tw}(\mathscr{C})_n \cong N\mathscr{C}_{2n+1}$. An element of $N\mathscr{C}_{2n+1}$ looks like a diagram of $2n+1$ composable morphisms between $2n$ objects of $\mathscr{C}$

$$\begin{array}{ccccccccc} \overline{0} & \longleftarrow & \overline{1} & \longleftarrow & \dots & \longleftarrow & \overline{n-1} & \longleftarrow & \overline{n} \\ \big\downarrow & & & & & & & & \\ 0 & \longrightarrow & 1 & \longrightarrow & \dots & \longrightarrow & n-1 & \longrightarrow & n \end{array}$$

but, composing arrows, this is the same as the diagram

$$\begin{array}{ccccccccc}
\overline{0} & \longleftarrow & \overline{1} & \longleftarrow & \dots & \longleftarrow & \overline{n-1} & \longleftarrow & \overline{n} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & 1 & \longrightarrow & \dots & \longrightarrow & n-1 & \longrightarrow & n
\end{array}$$

which is just an element of $N\,\mathrm{tw}(\mathscr{C})_n$. A quick inspection shows that this correspondence is compatible with the face and degeneracy maps as well. □

Now, the key result to use is one proved in [Seg73, Appendix 1], although he says it is "more or less due to Quillen."

Theorem 2.22. *For any $X \in \mathbf{sTop}$, there is a homeomorphism $|X| \cong |\mathrm{sd}(X)|$.*

Applying this to the twisted arrow category, we see that $B\mathscr{C} \cong B\,\mathrm{tw}(\mathscr{C})$. There are various other ways we can manipulate the data of $\mathscr{C}$ to get a new category— for example, the opposite category $\mathscr{C}^{\mathrm{op}}$, the over/under categories of an object $c \in \mathscr{C}$, and so on— and we can ask how these changes affect the resulting classifying space. In the case of $\mathscr{C}^{\mathrm{op}}$, it is perhaps unsurprising that $B\mathscr{C} \cong B\mathscr{C}^{\mathrm{op}}$, since geometric realization does not really care about the direction of the arrows. As for the over (or under) category of $c \in \mathscr{C}$, its classifying space will consist of the collection of n-simplices which all have $c \times \Delta^0$ as a vertex. In particular, the classifying space of an over (or under) category will be contractible, since it has an initial (or terminal) object (see Remark 3.12).

3 Topological Categories

If our category $\mathscr{C}$ comes with some topology in some sense, then we would want its classifying space to keep track of this information. For our purposes, the idea of a category "coming with a topology" will take the form of *categories internal to* **Top**. In this context, the nerve $N\mathscr{C}$ will be a simplicial space, and we can still talk about the classifying space $B\mathscr{C}$ as the realization of this simplicial space as we saw in Subsection 2.3. We will refer to such categories as *topological categories*, although it is worth noting this term is slightly overloaded in the literature. For instance, being internal to **Top** is distinct from being enriched[5] over **Top**, and the former is a stronger condition in this context as we shall discuss in Subsection 3.2.

3.1 What is a topological category?

The idea of an internal topological category is that both the objects and morphisms come with a topology, and we ask the categorical structure to respect this topology.

Definition 3.1. A small category $\mathscr{C}$ is a *topological category* (in the sense of being internal to **Top**) if it has a space of objects $\mathscr{C}_0$ and a space of morphisms $\mathscr{C}_1$ with four continuous structure maps:

$$\begin{array}{c} \mathscr{C}_0 \\ \text{dom} \uparrow \quad \downarrow i \quad \uparrow \text{cod} \\ \mathscr{C}_1 \end{array}, \qquad \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \xrightarrow{\ \circ\ } \mathscr{C}_1 \ .$$

- The *domain map* $\text{dom}\colon (f\colon X \to Y) \mapsto X$,
- The *codomain map* $\text{cod}\colon (f\colon X \to Y) \mapsto Y$,
- The *identity map* $i\colon X \mapsto \text{id}_X$,
- The *composition map* $\circ$ sends a pair of morphisms (f, g) to their composite $g \circ f = gf$. Here $\circ$ is defined on the pullback of $\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1$:

$$\begin{array}{ccc} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 & \xrightarrow{\pi_2} & \mathscr{C}_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \text{cod}} \\ \mathscr{C}_1 & \xrightarrow[\text{dom}]{} & \mathscr{C}_0 \end{array} .$$

These maps must satisfy a variety of compatibility conditions, expressed as diagrams in **Top**:

[5] It is also worth mentioning that both of these notions are distinct from topologically concrete categories (as in [rAHS04, §VI.21]).

$$
\begin{array}{ccc}
\mathscr{C}_0 & \xrightarrow{\;i\;} & \mathscr{C}_1 \\
& \diagdown\!\!\diagdown & {\scriptstyle \mathrm{dom}}\downarrow\downarrow{\scriptstyle \mathrm{cod}} \\
& & \mathscr{C}_0
\end{array}
,\qquad
\begin{array}{ccccc}
\mathscr{C}_1 & \xleftarrow{\;\pi_1\;} & \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 & \xrightarrow{\;\pi_2\;} & \mathscr{C}_1 \\
{\scriptstyle \mathrm{cod}}\downarrow & & \downarrow{\scriptstyle \circ} & & \downarrow{\scriptstyle \mathrm{dom}} \\
\mathscr{C}_0 & \xleftarrow[\mathrm{cod}]{} & \mathscr{C}_1 & \xrightarrow[\mathrm{dom}]{} & \mathscr{C}_0
\end{array}
.
$$

These two diagrams specify the domain and codomain of the identity map i and the composition map $\circ$, respectively. The following two diagrams assert that composition is unital (with identity i) and associative:

$$
\begin{array}{ccccc}
\mathscr{C}_0 \times_{\mathscr{C}_0} \mathscr{C}_1 & \xrightarrow{\;i \times \mathrm{id}\;} & \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 & \xleftarrow{\;\mathrm{id} \times i\;} & \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_0 \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \circ} & & \downarrow{\scriptstyle \pi_1} \\
\mathscr{C}_1 & = & \mathscr{C}_1 & = & \mathscr{C}_1
\end{array}
$$

$$
\begin{array}{ccc}
\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 & \xrightarrow{\;\circ \times \mathrm{id}\;} & \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \\
{\scriptstyle \mathrm{id} \times \circ}\downarrow & & \downarrow{\scriptstyle \circ} \\
\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 & \xrightarrow[\circ]{} & \mathscr{C}_1
\end{array}
.
$$

Of course, any small category is a topological category under the discrete topology, but there is often more than one way to do it. Many familiar notions from category theory have internal counterparts, such as functors and natural transformations.

Definition 3.2. A *continuous functor* is a map $F \colon \mathscr{C} \to \mathscr{D}$ between two topological categories that consists of two continuous maps,

$$F_0 \colon \mathscr{C}_0 \to \mathscr{D}_0 \quad \text{and} \quad F_1 \colon \mathscr{C}_1 \to \mathscr{D}_1,$$

which are compatible with the four structure maps. That is, such that the following diagrams commute:

$$
\begin{array}{ccc}
\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 & \xrightarrow{\;F_1 \times F_1\;} & \mathscr{D}_1 \times_{\mathscr{D}_0} \mathscr{D}_1 \\
{\scriptstyle \circ_{\mathscr{C}}}\downarrow & & \downarrow{\scriptstyle \circ_{\mathscr{D}}} \\
\mathscr{C}_1 & \xrightarrow[F_1]{} & \mathscr{D}_1
\end{array}
\qquad
\begin{array}{ccccc}
\mathscr{C}_1 & \overset{\mathrm{dom}}{\underset{\mathrm{cod}}{\rightrightarrows}} & \mathscr{C}_0 & \xrightarrow{\;i\;} & \mathscr{C}_1 \\
{\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle F_1} \\
\mathscr{D}_1 & \overset{\mathrm{dom}}{\underset{\mathrm{cod}}{\rightrightarrows}} & \mathscr{D}_0 & \xrightarrow[i]{} & \mathscr{D}_1
\end{array}
.
$$

We then assemble the category **Cat(Top)** whose objects are topological categories and whose morphisms are continuous functors.

Definition 3.3. A *continuous natural transformation* $\eta \colon F \to G$ between a pair of continuous functors $F, G \colon \mathscr{C} \rightrightarrows \mathscr{D}$ consists of a continuous map $\eta \colon \mathscr{C}_0 \to \mathscr{D}_1$ such that the following diagrams commute in **Top**:

$$\begin{array}{ccc} \mathscr{C}_0 & \xrightarrow{\eta} & \mathscr{D}_1 \\ & {}_F\searrow & \downarrow{\scriptstyle \mathrm{dom}} \\ & & \mathscr{D}_0 \end{array} \qquad \begin{array}{ccc} \mathscr{C}_0 & \xrightarrow{\eta} & \mathscr{D}_1 \\ & {}_G\searrow & \downarrow{\scriptstyle \mathrm{cod}} \\ & & \mathscr{D}_0 \end{array} .$$

That is, the map η assigns every $X \in \mathscr{C}_0$ a morphism $\eta_X \colon FX \to GX$. The naturality condition on η requires that the following diagram also commutes:

$$\begin{array}{ccc} \mathscr{C}_1 & \xrightarrow{(G,\ \eta \circ \mathrm{dom})} & \mathscr{D}_1 \times_{\mathscr{D}_0} \mathscr{D}_1 \\ {\scriptstyle (\eta \circ \mathrm{cod}, F)}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathscr{D}_1 \times_{\mathscr{D}_0} \mathscr{D}_1 & \xrightarrow[\circ]{} & \mathscr{D}_1 \end{array} .$$

Note that the commutativity of this diagram implies the usual naturality condition. That is, if $f \colon X \to Y \in \mathscr{C}_1$, the diagram above maps

$$\begin{array}{ccc} f & \longmapsto & (Gf, \eta_X) \\ \downarrow & & \downarrow \\ (\eta_Y, Ff) & \longmapsto & \eta_Y \circ Ff = Gf \circ \eta_X \end{array}$$

which is precisely the desired condition.

Remark 3.4. It is equivalent to require that η is a continuous functor

$$\eta \colon \mathscr{C} \times [1] \to \mathscr{D}$$

such that $\eta(-,0) = F$ and $\eta(-,1) = G$. Here $[1]$ is the poset category $0 \to 1$.

To see one direction, first suppose we have such a functor, so we get two maps $\eta_0 \colon \mathscr{C}_0 \times \{0,1\} \to \mathscr{D}_0$ and $\eta_1 \colon \mathscr{C}_1 \times \{\mathrm{id}_0, \to, \mathrm{id}_1\} \to \mathscr{D}_1$. We can define the natural transformation η by taking $\eta \colon \mathscr{C}_0 \to \mathscr{D}_1$ to be $\eta(c) = \eta_1(\mathrm{id}_c, \to)$; this is continuous because η_1 and the identity map i are both continuous. We need to check the relevant diagrams commute. First, we need to see that $\eta(c)$ maps Fc to Gc. This follows from compatibility of the functor with dom and cod (the left half of the second diagram in the definition). In particular, if $f \colon c \to c'$ in $\mathscr{C}$, then functoriality implies

$$\begin{array}{ccc} (f, \to) & \xrightarrow{\mathrm{dom}} & (c, 0) \\ {\scriptstyle \eta_1}\downarrow & & \downarrow{\scriptstyle \eta_0} \\ \eta_1(f, \to) & \xrightarrow[\mathrm{dom}]{} & \mathrm{dom}(\eta_1(f \to)) = Fc \end{array} \quad \text{and} \quad \begin{array}{ccc} (f, \to) & \xrightarrow{\mathrm{cod}} & (c', 1) \\ {\scriptstyle \eta_1}\downarrow & & \downarrow{\scriptstyle \eta_0} \\ \eta_1(f, \to) & \xrightarrow[\mathrm{cod}]{} & \mathrm{cod}(\eta_1(f \to)) = Gc' \end{array} .$$

We also need to check the naturality square, in particular that $\eta(c') \circ Ff = Gf \circ \eta(c)$. This follows from two applications of the diagram which encodes compatibility of the functor with composition (the first diagram in the definition). Specifically, we have commutative diagrams

$$\begin{array}{ccc} (f, \mathrm{id}_1) \times (\mathrm{id}_c, \to) & \xrightarrow{\eta_1 \times \eta_1} & \eta_1(f, \mathrm{id}_1) \times \eta_1(\mathrm{id}_c, \to) = (Gf, \eta(c)) \\ \downarrow{\scriptstyle \circ_{\mathscr{C} \times [1]}} & & \downarrow{\scriptstyle \circ_{\mathscr{D}}} \\ (f \circ \mathrm{id}_c, \mathrm{id}_1 \circ \to) = (f, \to) & \xrightarrow[\eta_1]{} & \eta_1(f, \to) = Gf \circ \eta(c) \end{array}$$

and

$$\begin{array}{ccc} (\mathrm{id}_{c'}, \to) \times (f, \mathrm{id}_0) & \xrightarrow{\eta_1 \times \eta_1} & \eta_1(\mathrm{id}_{c'}, \to) \times \eta_1(f, \mathrm{id}_0) = (\eta(c'), Ff) \\ \downarrow{\scriptstyle \circ_{\mathscr{C} \times [1]}} & & \downarrow{\scriptstyle \circ_{\mathscr{D}}} \\ (\mathrm{id}_{c'} \circ f, \to \circ \mathrm{id}_0) = (f, \to) & \xrightarrow[\eta_1]{} & \eta_1(f, \to) = \eta(c') \circ Ff \end{array}$$

which shows $\eta(c') \circ Ff = \eta_1(f, \to) = Gf \circ \eta(c)$. The reverse implication follows from a similar unwinding of diagrams.

Definition 3.5. A *continuous equivalence* of categories internal to **Top** consists of two continuous functors $F \colon \mathscr{C} \leftrightarrows \mathscr{D} \colon G$ together with two continuous natural isomorphisms $\eta \colon \mathrm{id}_{\mathscr{C}} \to GF$ and $\epsilon \colon FG \to \mathrm{id}_{\mathscr{D}}$.

3.2 Comparing to topologically enriched categories

There is another common way to talk about a category coming with topological structure, which is topologically enriched categories.

Definition 3.6. A category $\mathscr{C}$ is *topologically enriched* if it has a set of objects $\mathrm{Ob}\,\mathscr{C}$ and for each pair of objects $c, d \in \mathscr{C}$, there is a space of morphisms $\mathrm{Hom}(c, d)$. We also require certain continuous structure maps:

- For each trio $c, c', d \in \mathrm{Ob}\,\mathscr{C}$, composition $\mathrm{Hom}(c, c') \times \mathrm{Hom}(c', d) \to \mathrm{Hom}(c, d)$ must be continuous.
- For each object $c \in \mathrm{Ob}\,\mathscr{C}$, there is a continuous map $*_c \colon * \to \mathrm{Hom}(c, c)$ which picks out the identity of c.

Moreover, these maps must satisfy the following compatibility diagrams:

$$\begin{array}{ccccc} & & \mathscr{C}(a,b) \times \mathscr{C}(b,c) \times \mathscr{C}(c,d) & & \\ & \swarrow{\scriptstyle \circ \times \mathrm{id}} & & {\scriptstyle \mathrm{id} \times \circ}\searrow & \\ \mathscr{C}(a,c) \times \mathscr{C}(c,d) & \xrightarrow[\circ]{} & \mathscr{C}(a,d) & \xleftarrow[\circ]{} & \mathscr{C}(a,b) \times \mathscr{C}(b,d) \end{array}$$

This diagram says that composition is associative, and the next one says it is unital.

$$\begin{array}{ccccc} \mathscr{C}(a,a)\times\mathscr{C}(a,b) & \xrightarrow{\circ} & \mathscr{C}(a,b) & \xleftarrow{\circ} & \mathscr{C}(a,b)\times\mathscr{C}(b,b) \\ {\scriptstyle *_a\times \mathrm{id}}\uparrow & \nearrow{\scriptstyle proj_2} & & \nwarrow{\scriptstyle proj_1} & \uparrow{\scriptstyle \mathrm{id}\times *_b} \\ *\times\mathscr{C}(a,b) & & & & \mathscr{C}(a,b)\times * \end{array}$$

A *topologically enriched functor* $F\colon \mathscr{C} \to \mathscr{D}$ between topologically enriched categories is just a functor such that $F_{c,c'}\colon \mathrm{Hom}(c,c') \to \mathrm{Hom}(Fc, Fc')$ is continuous for each pair of objects $c, c' \in \mathscr{C}$. The category of topologically enriched categories and their functors is denoted by $\mathbf{Cat_{Top}}$.

There are also topologically enriched versions of natural transformations, equivalences of categories, and so on. The idea is always just to ask that the maps respect the topological structure on the homspaces.

Remark 3.7. Both internalization and enrichment can be defined more generally. For internal categories, the ambient category is required to be finitely complete[6] (cf. [ML71, §XII.1]). For enriched categories, the ambient category must be monoidal. For an arbitrary ambient category $\mathscr{V}$, categories internal to $\mathscr{V}$ may not be comparable to categories enriched in $\mathscr{V}$. That is, it is not true in general that being internal is a stronger condition than being enriched, or vice versa. However, in the special case of **Top**, we can say something to this effect.

If we compare the definition (internal) topological categories (Definition 3.1) to that of topologically enriched categories, we can see that there are many similarities. In particular, we might guess that a topologically enriched category consists of the same data as a topological category with a discrete object space. We can make this comparison more precise via a forgetful-cofree adjunction.

Definition 3.8. Let $U\colon \mathbf{Cat(Top)} \to \mathbf{Cat_{Top}}$ send a topological category $\mathscr{C}$ to the enriched category $U\mathscr{C}$ whose object set is the underlying set of $\mathscr{C}_0$, and whose homspaces are

$$U\mathscr{C}(c,c') = \mathscr{C}_1(c,c') \subseteq \mathscr{C}_1,$$

where $\mathscr{C}_1(c,c') = \mathrm{cod}^{-1}(c) \cap \mathrm{dom}^{-1}(c')$ for each pair of objects $c, c' \in \mathscr{C}$. Composition on the homspaces in $U\mathscr{C}$ is given by restriction of the composition in $\mathscr{C}$, and $*_c\colon * \to \{c\} \xrightarrow{i} \mathscr{C}_1$ for each $c \in \mathscr{C}$. The diagrams for a topological category ensure that the diagrams for a topologically enriched category commute. The forgetful functor U sends a functor between topological categories to itself, now as a functor of topologically enriched categories.

Now we define the "co-free" (i.e. right-adjoint) functor $F\colon \mathbf{Cat_{Top}} \to \mathbf{Cat(Top)}$. We assume all our categories are small.

[6]A *finitely complete* category has all finite products, pullbacks, and a terminal object.

Definition 3.9. Let $\mathscr{C}$ be a topologically enriched category, and define $F\mathscr{C}$ to be the topological category with (discrete) object space $F\mathscr{C}_0 = \mathrm{Ob}\,\mathscr{C}$ and morphism space $F\mathscr{C}_1 = \coprod_{c,c' \in \mathscr{C}} \mathrm{Hom}(c,c')$. This ensures composition $\circ = \coprod_{c,d,d'} \circ$ is continuous, and the domain and codomain maps will be open since $\mathrm{dom}^{-1}(c) = \coprod_{c'} \mathrm{Hom}(c,c')$ and $\mathrm{cod}^{-1}(d') = \coprod_{c} \mathrm{Hom}(c,c')$ are both open in $F\mathscr{C}_1$. As for the identity, we can define it on objects by $c \to * \xrightarrow{*_c} \mathrm{Hom}(c,c)$, which will be continuous since the domain is discrete. Again, the diagrams from the topologically enriched definition ensure that the necessary diagrams in the internal definition commute. A topologically enriched functor $f\colon \mathscr{C} \to \mathscr{D}$ maps to a continuous functor $Ff\colon F\mathscr{C} \to F\mathscr{D}$ with f_0 equal to f on objects (now viewed as a map of discrete spaces) and $f_1 = \coprod_{c,c' \in \mathscr{C}} f_{c,c'}$.

Proposition 3.1. *Let $\mathscr{C}$ be an (internal) topological category and let $\mathscr{D}$ be a topologically enriched category. Then there is an adjunction $U \dashv F$, so there is a bijection*

$$\mathbf{Cat}_{\mathbf{Top}}(U\mathscr{C}, \mathscr{D}) \cong \mathbf{Cat}(\mathbf{Top})(\mathscr{C}, F\mathscr{D}).$$

Proof. We need to show there are continuous natural transformations $\eta\colon FU \Rightarrow \mathrm{id}_{\mathbf{Cat}(\mathbf{Top})}$ and $\varepsilon\colon UF \Rightarrow \mathrm{id}_{\mathbf{Cat}_{\mathbf{Top}}}$ which satisfy the triangle identities. A quick check verifies that $UF = \mathrm{id}$, so $\varepsilon = \mathrm{id}$ works.

Note that FU does not change the underlying set of objects or morphisms of a topological category, but just the topologies on them. Specifically, $FU\mathscr{C}_0$ is the discrete space $(\mathscr{C}_0)_{disc}$ and $FU\mathscr{C}_1$ is $\coprod_{c,c' \in \mathscr{C}} \mathrm{Hom}(c,c')$ where each $\mathrm{Hom}(c,c')$ also has its original topology as a subspace of $\mathscr{C}_1$. So we can define η by just taking a component $\eta_{\mathscr{C}}\colon FU\mathscr{C} \to \mathscr{C}$ to send each object and morphism to itself. We know $\eta_{\mathscr{C}}$ is continuous on objects since the domain is discrete, and it is continuous on morphisms because the topology on $FU\mathscr{C}_1$ is a refinement of the one on $\mathscr{C}_1$. A quick check verifies that η commutes with the structure maps and satisfies the triangle identities. ☐

Among other things, we can use this adjunction to think of nerves of enriched categories as simplicial spaces, rather than just simplicial sets. We note that we would get the same construction of an enriched nerve using the simplicial bar construction instead (see [Rie09, §3.1]), which can also be used to form enriched nerves more generally.

3.3 Classifying spaces

The *classifying space* of $\mathscr{C}$ is the geometric realization of the nerve of $\mathscr{C}$, or in mathematical notation $B\mathscr{C} = |N\mathscr{C}|$.

To form the nerve of a topological category, we need to think of the collection of n composable morphisms instead as a *space* instead of just a set. To this end, we form the n^{th} level of the nerve inductively as the iterated pullback with n factors,

$$N\mathscr{C}_n = \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \cdots \times_{\mathscr{C}_0} \mathscr{C}_1,$$

where each $\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1$ is the limit of $\mathscr{C}_1 \xrightarrow{\text{cod}} \mathscr{C}_0 \xleftarrow{\text{dom}} \mathscr{C}_1$. As in the simplicial set version of the nerve, $N\mathscr{C}_0 = \operatorname{Ob}\mathscr{C}$. A continuous functor $\mathscr{C} \to \mathscr{D}$ induces a map $N\mathscr{C} \to N\mathscr{D}$, and one can verify that we indeed get a functor $N\colon \mathbf{Cat}(\mathbf{Top}) \to \mathbf{sTop}$.

Composing the nerve and the geometric realization, we get the classifying space of a topological category. Segal's work in [Seg68] verifies that the classifying space map $B\colon \mathbf{Cat}(\mathbf{Top}) \to \mathbf{Top}$ retains its functorial properties, meaning that a continuous functor $F\colon \mathscr{C} \to \mathscr{D}$ induces a continuous map $BF\colon B\mathscr{C} \to B\mathscr{D}$, and moreover equivalent topological categories have homotopy equivalent classifying spaces. The latter fact is a consequence of the following theorem:

Theorem 3.10 ([Seg68])**.** *Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be continuous functors with a continuous natural transformation $\eta\colon F \to G$. Then the induced maps $BF, BG\colon B\mathscr{C} \to B\mathscr{D}$ are homotopic.*

Remark 3.11. We say a functor $F\colon \mathscr{C} \to \mathscr{D}$ has a *lax inverse* if there is a functor $F'\colon \mathcal{D} \to \mathcal{C}$ such that both FF' and $F'F$ have natural transformations to the relevant identity functors. There are no conditions imposed on the interactions of these natural transformations, so this notion is a strictly weaker than that of an adjunction, although adjoint pairs and equivalences of categories are examples of lax inverses. A corollary of the theorem above is that any functor which admits a lax inverse will realize to a homotopy equivalence.

Remark 3.12. Note that this theorem implies that if $\mathscr{C}$ has an initial (or terminal) object, then the classifying space $B\mathscr{C}$ is contractible, induced by the continuous natural transformation between $\mathrm{id}_{\mathscr{C}}$ and the constant functor on the initial (or terminal) object.

3.4 The Reedy model structure

There is a model structure on simplicial spaces, called the *Reedy model structure*, which is inherited from giving $\mathbf{\Delta}$ the structure of a Reedy category. For the purposes of these notes, we will focus on how this model structure affects nerves of topological categories, specifically, and we will largely ignore the model category theory (for instance, we will not define what a model category is). The reader is invited to see [Dug08, §13-14] and [RV14] for a discussion of Reedy categories and the Reedy model structure in a broader context.

3.4.1 Reedy cofibrancy

We might hope that a levelwise equivalence of simplicial spaces realizes to an equivalence of spaces, but unfortunately this is not always the case. That is, we can have a map of simplicial spaces $X \to Y$ such that each component $X_n \to Y_n$ is a homotopy equivalence, but $|X|$ and $|Y|$ are not homotopy equivalent. This is essentially because colimits generally fail to preserve equivalences, as we will see later in Example 3.16. One way to fix this issue is to use something like the homotopy colimit, called the *fat realization* of X (see [Dug08, Remark 3.6]), $||X||$, which is essentially the geometric realization without collapsing the degeneracies. Another approach is to require certain maps in X to be cofibrations; this is known as *Reedy cofibrancy*.

Definition 3.13. A simplicial space X is *Reedy cofibrant* (or *proper*) if every latching map $L_nX \hookrightarrow X_n$ is a cofibration, where

$$L_nX = \bigcup_{i=0}^{n-1} s_i(X_{n-1})$$

is the n^{th} *latching object*.

We can think of $L_nX \subseteq X_n$ as the set of degenerate n-simplices, which gives a natural inclusion $L_nX \hookrightarrow X_n$ (this is the n^{th} latching map in the definition above). One sufficient condition for being Reedy cofibrant is that every degeneracy map $s_j\colon X_n \to X_{n+1}$ is a cofibration (this condition is sometimes called being *good*). It turns out that this restriction gives us the kind of homotopy invariance that we want.

Theorem 3.14. *Let $f\colon X \to Y$ be a map of Reedy cofibrant simplicial spaces. If $f_n\colon X_n \to Y_n$ is a homotopy equivalence for all n, then $|f|\colon |X| \to |Y|$ is a homotopy equivalence.*

Our statement of this result follows [May72, Chapter 11] and [Seg74, Appendix A], although the reader familiar with model categories may be interested in the approach of [RV14] (see their Corollary 10.6 for a statement of the theorem above).

Remark 3.15. Sometimes the notion of being Reedy cofibrant or good requires the relevant maps to be *closed* cofibrations, but for sufficiently nice spaces (e.g. Hausdorff) every cofibration is a closed inclusion, so it suffices to just ask for cofibrations.

Recall from Example 2.8 that we can replace any diagram $D\colon I \to \mathbf{Top}$ with a simplicial space $srep(D)$. For nice enough diagrams (e.g. diagrams which land in CW complexes), the simplicial replacement is Reedy cofibrant. So an equivalence between two nice diagrams D and D' will induce an equivalence on their simplicial replacements, and the theorem above implies $|srep(D)| \simeq |srep(D')|$.

In fact, this result is true for any two equivalent diagrams $D, D'\colon I \to \mathbf{Top}$. This is because the realization of $srep(D)$ is a model for the *homotopy colimit* of a diagram $D\colon I \to \mathbf{Top}$, written $\operatorname{hocolim} D$, which is like the colimit of D "up to homotopy." If taking a colimit is like gluing spaces together, then taking the homotopy colimit is like gluing spaces together with extra wiggle room. For more on homotopy co/limits, see [Dug08, §1.4–5] or [Hir14]. One nice property of homotopy colimits is that they preserve weak equivalences, which is not true in general for colimits.

Example 3.16. Consider the indexing category $I = \bullet \leftarrow \bullet \rightarrow \bullet$ and let D be the diagram $*_N \leftarrow S^{n-1} \rightarrow *_S$. (Both $*_N$ and $*_S$ are the trivial space, but we give them different names so we can tell them apart in our computation.) The colimit of D is just a point, but we will show that the homotopy colimit[7] is the suspension $S(S^{n-1}) \cong S^n$, which is not contractible, i.e. the colimit and homotopy colimit are not equivalent!

Let's compute the homotopy colimit of D using the simplicial replacement. Since there are no interesting ways to compose two or more morphisms, $srep(D)_n$ contains only degenerate simplices for $n \geq 2$. This means we only need to think about levels 0 and 1 to compute the homotopy colimit, since all the higher levels will be collapsed in the realization. The non-degenerate simplices in $srep(D)_0$ are $*_N$, S^{n-1}, and $*_S$, and the non-degenerate simplices in $srep(D)_1$ are S^{n-1}_N and S^{n-1}_S, which are associated to the maps $S^{n-1} \rightarrow *_N$ and $S^{n-1} \rightarrow *_S$, respectively. Then we can write

$$|srep(D)| = (*_N \amalg S^{n-1} \amalg *_S) \times \Delta^0 \coprod (S^{n-1}_N \amalg S^{n-1}_S) \times \Delta^1 / \sim$$

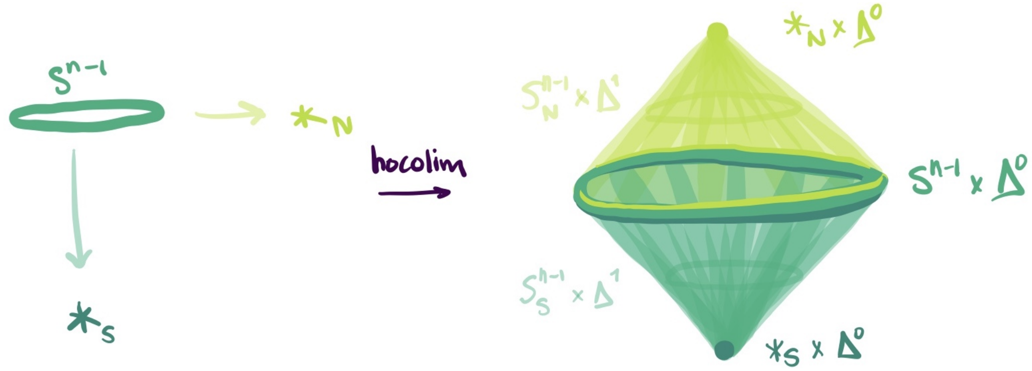
where $S^{n-1} \times \Delta^1$ is glued to $*_N \times \Delta^0$ and $S^{n-1} \times \Delta^0$ via

$$(x, 1) \sim (*_N, *) \quad \text{and} \quad (x, 0) \sim (x, *)$$

for $x \in S^{n-1}_N$; similarly, S^{n-1}_S is glued to $*_S \times \Delta^0$ and $S^{n-1} \times \Delta^0$ by

$$(x, 0) \sim (*_S, *) \quad \text{and} \quad (x, 1) \sim (x, *)$$

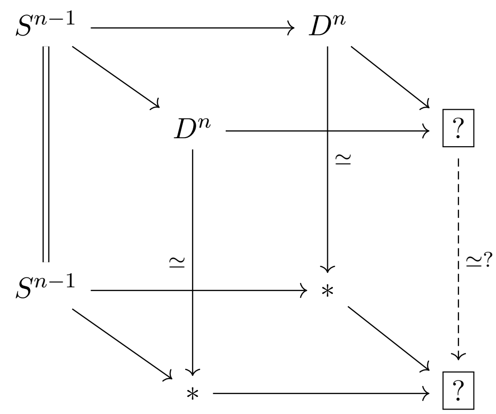
for $x \in S^{n-1}_S$. This describes $|srep(D)|$ as the suspension of S^{n-1}, and hence $|srep(D)| \cong S^{n-1}$.



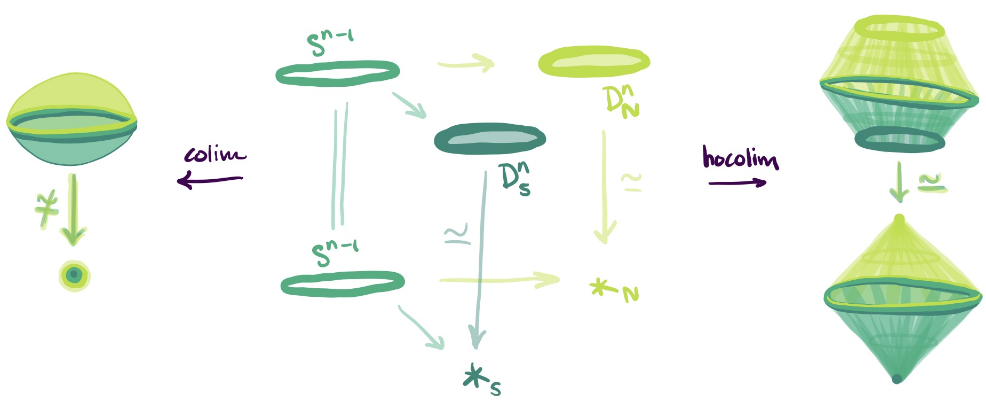
Now let's see an example of how the homotopy colimit preserves equivalences. Take D' to be the diagram $D^n \leftarrow S^{n-1} \rightarrow D^n$. We have equivalences $D^n \rightarrow *$ which give

[7] For this special indexing diagram, the homotopy colimit is usually called the *homotopy pushout*.

us the following diagram:



If we take the colimit, then the top ? becomes S^n but the bottom ? becomes $*$, which are not equivalent. However, if we take the homotopy colimit instead, then we *will* get an induced equivalence.



An interesting observation is that, in this case, colim $D' \simeq$ hocolim D'; this is because the inclusion $S^{n-1} \to D^n$ is a cofibration. In general, if enough of the maps in the diagram are cofibrations, then the usual colimit models the homotopy colimit (cf. [Hir14, §10.2]).

In the context of classifying spaces, we can reformulate Reedy cofibrancy of $N\mathscr{C}$ in terms of the identity $i\colon \mathscr{C}_0 \to \mathscr{C}_1$, following [Rob02], since the degenerate simplicies all come from i.

Definition 3.17. A topological category is *well-pointed* if the inclusion $i\colon \mathscr{C}_0 \hookrightarrow \mathscr{C}_1$ makes $(\mathscr{C}_1, \mathscr{C}_0)$ an NDR-pair over $\mathscr{C}_0$, where $\mathscr{C}_1$ is considered as a space over $\mathscr{C}_0$ via both the domain and codomain maps (and $\mathscr{C}_0$ is a space over itself via the identity).

This means there are maps

$$\begin{array}{ccc} \mathscr{C}_1 & \xrightarrow{u} & I \times \mathscr{C}_0 \\ & \underset{\text{dom}}{\searrow} & \downarrow \text{proj}_2 \\ & & \mathscr{C}_0 \end{array} \qquad \begin{array}{ccc} \mathscr{C}_1 \times I & \xrightarrow{h} & \mathscr{C}_1 \\ & \searrow & \downarrow \text{dom} \\ & & \mathscr{C}_0 \end{array}$$

so that

(i) $\mathscr{C}_0 = u^{-1}(\{0\} \times \mathscr{C}_0)$,

(ii) $h_0 = \text{id}_{\mathscr{C}_1}$ and $h|_{\mathscr{C}_0 \times I} = \text{proj}_{\mathscr{C}_0}$,

(iii) $h_1(x) \in \mathscr{C}_0$ for $x \in u^{-1}[0, 1)$

as maps over $\mathscr{C}_0$, and there are also similarly defined maps when we replace dom with cod.

The idea is that $(\mathscr{C}_1, \mathscr{C}_0)$ is an NDR pair in such a way that the relevant homotopies do not move $\mathscr{C}_0$ when viewed as the subspace of domains (or codomains). In particular, $i\colon \mathscr{C}_0 \to \mathscr{C}_1$ must be a (closed) cofibration.

Proposition 3.2. *The nerve of a well-pointed category is good.*

Proof. The key fact to use is that NDR pairs over a space are preserved under pullback; in particular, if $X \to \mathscr{C}_0$ is any space over $\mathscr{C}_0$, then the fact that $(\mathscr{C}_1, \mathscr{C}_0)$ is NDR over $\mathscr{C}_0$ implies that $(\mathscr{C}_1 \times_{\mathscr{C}_0} X, \mathscr{C}_0 \times_{\mathscr{C}_0} X)$ is NDR over $\mathscr{C}_0$. Moreover, since we assumed $(\mathscr{C}_1, \mathscr{C}_0)$ is NDR over $\mathscr{C}_0$ in two ways, for $\mathscr{C}_1$ as a space over $\mathscr{C}_0$ via $\text{dom}\colon \mathscr{C}_1 \to \mathscr{C}_0$ *and* $\text{cod}\colon \mathscr{C}_1 \to \mathscr{C}_0$, we can consider the limit over

$$Y \longrightarrow \mathscr{C}_0 \xleftarrow{\text{dom}} \mathscr{C}_1 \xrightarrow{\text{cod}} \mathscr{C}_0 \longleftarrow X$$

for any spaces Y, X over $\mathscr{C}_0$ and conclude that the pair $(Y \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} X, Y \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} X)$ is NDR over $\mathscr{C}_0$. Now, applying this to the degeneracy maps

$$N_n\mathscr{C} \simeq N_j\mathscr{C} \times_{\mathscr{C}_0} \mathscr{C}_0 \times_{\mathscr{C}_0} N_{n-j}\mathscr{C} \xrightarrow{s_j} N_j\mathscr{C} \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} N_{n-j}\mathscr{C} \simeq N_{n+1}\mathscr{C}$$

which insert an identity, we get that $s_j\colon N_n\mathscr{C} \to N_{n+1}\mathscr{C}$ are cofibrations for all $j = 0, \dots, n$. □

Remark 3.18. If $\mathscr{C}$ is an internal category arising from an enriched category as in Subsection 3.2, then we know $\mathscr{C}_0$ is discrete and $\mathscr{C}_1 = \coprod_{c,c' \in \mathscr{C}} \text{Hom}(c, c')$. Assuming each $c \in \mathscr{C}$ has no non-trivial automorphisms, the inclusion of the summands $i(\mathscr{C}_0) = \coprod_{c \in \mathscr{C}} \text{Hom}(c, c)$ into $\mathscr{C}_1$ is a closed cofibration. Hence the nerve of such a category is Reedy cofibrant.

In light of Theorem 3.14, we can ask whether there is something weaker than an equivalence of categories which would induce an equivalence of classifying spaces of well-pointed categories. For instance, given a levelwise equivalence $\mathscr{C}_0 \simeq \mathscr{D}_0$ and $\mathscr{C}_1 \simeq \mathscr{D}_1$, do we get $B\mathscr{C} \simeq B\mathscr{D}$? In order to get a levelwise equivalence of nerves, we need this to come from a continuous functor $F\colon \mathscr{C} \to \mathscr{D}$ which induces a levelwise equivalence $NF\colon N\mathscr{C} \to N\mathscr{D}$. This is true if

$$\begin{array}{ccc} N_n\mathscr{C} & \longrightarrow & \mathscr{C}_1 \\ \downarrow & & \downarrow{\scriptstyle \mathrm{dom}} \\ N_{n-1}\mathscr{C} & \xrightarrow[\mathrm{cod}]{} & \mathscr{C}_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} N_n\mathscr{D} & \longrightarrow & \mathscr{D}_1 \\ \downarrow & & \downarrow{\scriptstyle \mathrm{dom}} \\ N_{n-1}\mathscr{D} & \xrightarrow[\mathrm{cod}]{} & \mathscr{D}_0 \end{array}$$

are homotopy pullbacks[8] for all $n \geq 2$, since the homotopy invariance of pullbacks will then imply $N_nF\colon N_n\mathscr{C} \xrightarrow{\sim} N_n\mathscr{D}$ for all $n \geq 0$. (We could also swap the role of the domain and codomain maps if preferred.) Thus we see that it is sufficient in some cases to merely require that $F\colon \mathscr{C} \to \mathscr{D}$ is a levelwise equivalence in order to conclude $BF\colon B\mathscr{C} \simeq B\mathscr{D}$.

Remark 3.19. The strict pullbacks $N_n\mathscr{C}$, $N_n\mathscr{D}$ are homotopy pullbacks if either the domain (source) or codomain (target) map is a fibration in both $\mathscr{C}$ and $\mathscr{D}$. This is true, for example, if $\mathscr{C}$ and $\mathscr{D}$ have discrete object space.

Remark 3.20. In many cases, the levelwise equivalences $\mathscr{C}_0 \simeq \mathscr{D}_0$ and $\mathscr{C}_1 \simeq \mathscr{D}_1$ coming from $F\colon \mathscr{C} \to \mathscr{D}$ will likely be witnessed by a lax inverse $G\colon \mathscr{D} \to \mathscr{C}$, in which case Remark 3.11 tells us we get an equivalence of classifying spaces. The discussion above is meant to explore the situation where there may not exist such a G.

3.4.2 Reedy fibrancy

The dual notion to Reedy cofibrant is appropriately the named condition of *Reedy fibrancy*. The machinery of Reedy co/fibrancy works for more general *Reedy categories*, but we will focus specifically on $\mathbf{\Delta}^{\mathrm{op}}$ (which is the prototypical example of a Reedy category). Reedy fibrancy is also related to the construction of *homotopy limits*, the dual notion to homotopy colimits we discussed in Example 3.16 (see [Dug08, §5] for details).

Definition 3.21. A simplicial space X is *Reedy fibrant* if the *matching maps* $X_n \to M_nX$ are fibrations, where M_nX is the nth *matching object*

$$M_nX := \lim_{\substack{[n]\to[j] \\ j<n}} X$$

[8]Technically, we mean $N\mathscr{C}_n \xrightarrow{\mathrm{cod}} \mathscr{C}_0$ to be $N\mathscr{C}_n \xrightarrow{\circ} \mathscr{C}_1 \xrightarrow{\mathrm{cod}} \mathscr{C}_0$ where $\circ$ denotes the composition of the n morphisms from $N\mathscr{C}_n$. We omit this finer detail for the sake of simplicity.

The definition of the matching object can appear a bit unwieldy at first, so it may often be easier to deal with another, more concrete description discussed in [RV14, Examples 3.14 and 3.22]. In particular, the elements of M_nX can be described as maps $\partial\Delta^n \to X$, where $\partial\Delta^n$ is the simplicial set generated by Δ^n without its unique non-degenerate n-simplex, and the matching map $X_n \to M_nX$ is the one which sends a n-simplex to its boundary. Another way to say this is that M_nX is the n-simplices of the $(n-1)$-coskeleton[9] of X.

Example 3.22. Recall that Δ^n is the simplicial set with $\Delta^n_k = \mathbf{\Delta}([k],[n])$. The k^{th} matching object $M_k\Delta^n$ consists of maps $\partial\Delta^k \to \Delta^n$. But every map $\partial\Delta^k \to \Delta^n$ has a unique filler $\Delta^k \to \Delta^n$ determined by $\partial{\Delta^k}_{k-1} \to \Delta^n_{k-1}$, and hence $M_k\Delta^n$ can be identified with Δ^n_k. So in this case the matching map is just the identity, so Δ^n is Reedy fibrant.

We have introduced Reedy fibrancy of simplicial spaces so that we can discuss a remarkable result from [WD95]. Recall that every simplicial space X has an underlying simplicial set X^δ, where we've given everything the discrete topology, and we have a continuous map of simplicial spaces $X^\delta \to X$. It turns out that sometimes the topology on X doesn't really matter after realization: [WD95, Proposition 4.6] tells us that this map realizes to a weak equivalence when X has nice enough properties. We note that their result is stated for *based* simplicial spaces but the proof translates basically verbatim to unbased simplicial spaces.

Theorem 3.23. *Suppose X is a Reedy fibrant and Reedy cofibrant simplicial space. Then $|X^\delta| \to |X|$ is a weak equivalence.*

The proof of this result uses the bisimplicial space $Z_{*,*} = S_*(X_*)$ (where S_* is the total singular complex from Example 2.3) so $Z_{k,*} = S_k(X_*)$ and $Z_{*,n} = S_*(X_n)$. The advantage of using this bisimplicial space is that it gives us a way to translate between X^δ and X, because $S_0(X_*) = X^\delta$ and $|S_*(X_n)|$ is weakly equivalent to X_n for each $n \geq 0$. We then have

$$|X^\delta| = |S_0(X_*)| \xrightarrow{(1)} |[k] \mapsto |S_k(X_*)|| \cong |diag(S_*(X_*))| \cong |[n] \mapsto |S_*(X_n)|| \xrightarrow{(2)} |X|.$$

The fact that (1) and (2) are weak equivalences follows from Reedy fibrancy and cofibrancy of X_*, respectively.

Using this theorem, we can investigate when the topology on the category does not affect the resulting classifying space, at least up to weak equivalence. That is, if we let $\mathscr{C}^\delta$ denote the "discretized" version of $\mathscr{C}$, we can ask when $B\mathscr{C}$ and $B(\mathscr{C}^\delta)$ are equivalent. Because $N(\mathscr{C}^\delta)$ is the same as the simplicial set underlying $N\mathscr{C}$, we can make use of Theorem 3.23. Specifically, if the nerve $N\mathscr{C}$ is Reedy cofibrant *and* Reedy fibrant, then Theorem 3.23 tells us that the map $B(\mathscr{C}^\delta) \to B(\mathscr{C})$ is a weak

[9]The k-coskeleton of a simplical space X has the same m-simplicies as X for $m \leq k$ and produces a simplex of degree $m > k$ whenever there is a compatible family of m-faces.

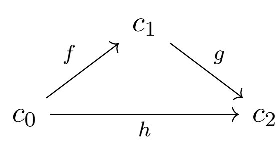
equivalence. This motivates us to characterize Reedy fibrancy of the nerve in terms of properties of $\mathscr{C}$.

The first thing to notice is that the matching maps turn out to be identities for $n \geq 3$. In particular, the nth matching object

$$M_n N\mathscr{C} = \lim_{\substack{[n]\to[k] \\ k<n}} N\mathscr{C}$$

is precisely $\mathscr{C}_1 \times_{\mathscr{C}_0} \cdots \times_{\mathscr{C}_0} \mathscr{C}_1 = N\mathscr{C}_n$ for $n \geq 3$. (This is because the nerve is 2-coskeletal, see [Rie14, Example 1.2.10].) For $n \leq 2$, we can make the conditions explicit:

- When $n = 0$, the matching object is the terminal object, so we are just asking if $N\mathscr{C}_0 = \mathscr{C}_0$ is fibrant (and all spaces are fibrant in the classical model structure).
- When $n = 1$, recall that $\partial\Delta^1 = \Delta^0 \amalg \Delta^0$, so an element $(\partial\Delta^1 \to N\mathscr{C}) \in M_1 N\mathscr{C}$ picks out two 0-simplices of $N\mathscr{C}$. That is, we can identify $M_1 N\mathscr{C}$ with $\mathscr{C}_0 \times \mathscr{C}_0$, and the matching map sends a morphism $f \in \mathscr{C}_1$ to $(\mathrm{dom}(f), \mathrm{cod}(f))$. So in order for $N\mathscr{C}$ to be Reedy fibrant, we need this map to be a fibration.
- For $n = 2$, we are looking at $(\partial\Delta^2 \to N\mathscr{C}) \in M_2 N\mathscr{C}$ which picks out a triangle of morphisms

  where the triangle *does not necessarily commute*. We asking for the inclusion of commuting triangles into all triangles to be a fibration onto its image. Note that if $N_2\mathscr{C} \to M_2 N\mathscr{C}$ is a fibration, then $\circ\colon N_2\mathscr{C} \to \mathscr{C}_1$ must be as well since we can post-compose with the projection $M_2 N\mathscr{C} \to \mathscr{C}_1$ onto the third coordinate, and this map admits a section which factors through $N_2\mathscr{C}$.

If these conditions hold for a well-pointed category (see Definition 3.17), so $N\mathscr{C}$ is Reedy fibrant and Reedy cofibrant, then the continuous functor $\mathscr{C}^\delta \to \mathscr{C}$ induces a weak equivalence on classifying spaces. Reedy fibrancy of nerves is a somewhat rare property, since Reedy cofibrancy is a reasonable (and desirable) property but $B\mathscr{C}$ typically has more complicated topology than $B\mathscr{C}^\delta$.

Remark 3.24. Note that a necessary condition for $\mathscr{C}$ to have a Reedy fibrant and cofibrant nerve is that the structure maps $(\mathrm{dom}, \mathrm{cod})\colon \mathscr{C}_1 \to \mathscr{C}_0 \times \mathscr{C}_0$ and $\circ\colon \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \to \mathscr{C}_1$ are fibrations and $i\colon \mathscr{C}_0 \to \mathscr{C}_1$ is a cofibration.

3.5 Generalizing Quillen's Theorems for Topological Categories

Quillen's Theorems A and B (Subsection 2.4) for ordinary categories are extremely useful, since they tell us when taking classifying spaces preserves homotopy fibers, in some sense. In the setting of topological categories the situation is somewhat more complicated, primarily because our nerves are now simplicial spaces instead of simplicial sets (and spaces are generally more delicate than sets). However, after some careful thought, we can find suitable reformulations within this topological context. Our exposition of Theorem A follows [Rob02], and we derive a version of Theorem B from [ERW19]. A more general version of Theorem B also appears in [Mey85]. Finally, we apply a topological version of Thomason's homotopy colimit theorem [Tho79] to deduce that cofibers of maps of classifying spaces may always be modeled as the classifying space of a certain Grothendieck construction. We assume all our categories are well pointed so the nerves are Reedy cofibrant.

3.5.1 Topological Theorem A

If we want to generalize Quillen's Theorem A (Theorem 2.14) to topological categories, a good first strategy is to try and mimic as much of the original proof as possible, and observe what goes wrong. Recall that the original proof used a category $\mathscr{S}(F)$ whose nerve looked like $(d_0 \to \dots d_n \to F(c_0), c_0 \to \cdots \to c_n)$, and the diagonal of this bisimplicial set looked like the nerve of a category $\mathscr{S}(F)$. In the topological setting, we can form $\mathscr{S}(F)$ as a *topological* category by forming it as a pullback

$$\begin{array}{ccc} \mathscr{S}(F) & \longrightarrow & \mathscr{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle F} \\ \operatorname{tw}\mathscr{D} & \xrightarrow[\text{cod}]{} & \mathscr{D} \end{array}$$

That is, the object space is $\mathscr{D}_1 \times_{\mathscr{D}_0} \mathscr{C}_0$ and morphism space $N_2\mathscr{D} \times_{\mathscr{D}_0} \times \mathscr{C}_1$. This category has the same objects and morphisms as the old $\mathscr{S}(F)$ from Theorem A, just with a topology. We again have a diagram

$$\begin{array}{ccccc} \mathscr{C} & \longleftarrow & \mathscr{S}(F) & \longrightarrow & \mathscr{D}^{\mathrm{op}} \\ \big\downarrow{\scriptstyle F} & & \big\downarrow & & \big\| \\ \mathscr{D} & \xleftarrow[\sim]{} & \mathscr{S}(\mathrm{id}_{\mathscr{D}}) = \operatorname{tw}\mathscr{D} & \xrightarrow[\sim]{} & \mathscr{D}^{\mathrm{op}} \end{array} .$$

We need to show that the top horizontal arrows are equivalences. Let's start with the right one. In the original proof, we noticed that the map $N_n\mathscr{S}(F) \to N_n\mathscr{D}$ which sends $(d_0 \to \cdots \to d_n \to F(c_0), c_0 \to \cdots \to c_n) \mapsto d_0 \to \cdots \to d_n$ is really a map

$$\coprod_{d_0 \to \cdots \to d_n} d{\downarrow}F \to \coprod_{d_0 \to \cdots \to d_n} *.$$

Then, by the assumption that all the $d \downarrow F$ were contractible, we have a levelwise equivalence of simplicial sets and therefore we get an equivalence after realization.

Now, the trouble for topological categories is we can no longer identify $N_n\mathscr{D}$ with $\coprod_{d_0 \to \cdots \to d_n} *$. However, upon careful reflection, we can come up with an analogous argument which works for topological categories. The key observation is that the collection of "fibers" $d \downarrow F$ now has a topology, coming from the topology of the objects $\mathscr{D}_0$. (This can be visualized like a "fiber bundle" living over $\mathscr{D}_0$). We can formalize this idea as a topological category $\mathscr{D}_0 \downarrow F$ which is defined as a pullback over a kind of "tangent" category.

Definition 3.25. Define $T\mathscr{D}$ as the pullback

$$\begin{array}{ccc} T\mathscr{D} & \longrightarrow & ar(\mathscr{D}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{dom}} \\ \mathscr{D}_0 & \xrightarrow[i]{} & \mathscr{D} \end{array},$$

where $\mathscr{D}_0$ in the diagram is the category with object space $\mathscr{D}_0$ and only identity morphisms (so $B\mathscr{D}_0 \cong D_0$). More explicitly, objects of $T\mathscr{D}$ are morphisms $d_0 \to d$ in $\mathscr{D}$ and morphisms are commuting triangles

$$\begin{array}{ccc} d_0 & \longrightarrow & d \\ & \searrow & \downarrow \\ & & d' \end{array}.$$

This can be thought of as the collection of based "path spaces" over all basepoints unioned over the object space $\mathscr{D}_0$. If $\mathscr{D}$ is discrete, then this is just a disjoint union. We will form $\mathscr{D}_0 \downarrow F$ as a the part of $T\mathscr{D}$ where the targets of the morphisms all land in the image of F.

Definition 3.26. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a continuous functor between topological categories. The category $\mathscr{D}_0 \downarrow F$ is the pullback

$$\begin{array}{ccc} \mathscr{D}_0 \downarrow F & \longrightarrow & \mathscr{C} \\ \downarrow & & \downarrow{\scriptstyle F} \\ \mathscr{D}_0 & \xrightarrow[i]{} & \mathscr{D} \end{array}.$$

The objects are morphisms $d \to F(c)$ and the morphisms are commuting triangles

$$\begin{array}{ccc} d & \longrightarrow & F(c_0) \\ & \searrow & \downarrow{\scriptstyle F(f)} \\ & & F(c_1) \end{array}$$

where $f\colon c_0 \to c_1$ in $\mathscr{C}$.

Note that when $\mathscr{D}$ is discrete, $\mathscr{D}_0 \downarrow F$ is the disjoint union of the fibers $d \downarrow F$. In the discrete case, we asked for each of these fibers to be contractible, which is like asking for $\mathscr{D}_0 \downarrow F$ to be equivalent to $\mathscr{D}_0$. However, we need something a bit stronger: we need $\mathscr{D}_0 \downarrow F$ to contract down to $\mathscr{D}_0$ without moving the basepoints $\mathscr{D}_0$. This condition can be summarized as a certain map being *shrinkable*.

Definition 3.27. A map $p\colon X \to Y$ in **Top** is *shrinkable* if it admits a section $s\colon Y \to X$ so that $sp\colon X \to X$ is homotopic to the identity in the category $\mathbf{Top}/Y$ (this is sometimes called being *fiberwise homotopic* to the identity).

For topological Theorem A, $\mathscr{D}_0 \downarrow F$ will play the role of $d \downarrow F$, and shrinkability of $\mathscr{D}_0 \downarrow F \to \mathscr{D}_0$ (where we mean the map is shrinkable after taking classifying spaces) replaces contractibility of the fiber $d \downarrow F$. In particular, $\mathscr{D}_0 \downarrow F \to \mathscr{D}_0$ must be a homotopy equivalence but in such a way that the "base space" of objects $\mathscr{D}_0$ is not disturbed. Of course if $\mathscr{D}_0$ is discrete, then it is enough to ask for each fiber to be contractible without any extra conditions.

Note that $N_n\mathscr{S}(F) = N_n\mathscr{D} \times_{\mathscr{D}_0} \mathscr{D}_0 \downarrow F$ and so if $\mathscr{D}_0 \downarrow F \to \mathscr{D}_0$ is shrinkable then so is $N_n\mathscr{D} \times_{\mathscr{D}} \mathscr{D}_0 \downarrow F \to N_n\mathscr{D} \times_{\mathscr{D}_0} \mathscr{D}_0$ (since pullbacks of shrinkable maps are shrinkable), and so in particular this map is a homotopy equivalence. But this is exactly the map $\mathscr{S}(F) \to \mathscr{D}^{\mathrm{op}}$, and thus we have shown this map is an equivalence.

Now for the functor $\mathscr{S}(F) \to \mathscr{C}$ which projects $(c, d \to Fc)$ to c. Running a similar sort of argument, we see that we need the map $B(\mathscr{D} \downarrow F(\mathscr{C}_0)) \to \mathscr{C}_0$ to be shrinkable, where $\mathscr{D} \downarrow F(\mathscr{C}_0)$ is a topological category (also defined as a pullback) whose objects look like $d \to F(c_0)$ and whose morphisms are triangles

$$\begin{array}{ccc} d & \longrightarrow & F(c_0) \\ \uparrow & \nearrow & \\ d' & & \end{array} .$$

This map is always shrinkable because the projection functor $\mathscr{D} \downarrow F(\mathscr{C}_0) \to \mathscr{C}_0$ has a lax inverse which sends each $c_0 \in \mathscr{C}$ to the terminal morphism $F(c_0) = F(c_0)$.

Theorem 3.28. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a continuous functor between well-pointed topological categories. If $\mathscr{D}_0 \downarrow F \to \mathscr{D}_0$ is shrinkable, then $F\colon \mathscr{C} \to \mathscr{D}$ is an equivalence.*

3.5.2 Topological Theorem B

In this section, we derive a version of Quillen Theorem B from the more general statement in [ERW19, Theorem 4.9]. The flavor of this result is a bit different than our version of Quillen Theorem A; it may be possible to use similar techniques as in the previous subsection to derive a different formulation of Quillen Theorem B (this would require, among other things, understanding what it means for $\mathscr{D}_0 \downarrow F \to \mathscr{D}_0$ to be a "quasifibration over $\mathscr{D}_0$").

Theorem 3.29. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a continuous functor between topological categories. Suppose that the following conditions hold:*

(I) *for all $d \to d' \in \mathscr{D}_1$, the induced map $B(F\downarrow d) \to B(F \downarrow d')$ is an equivalence,*

(II) *$\mathrm{Ob}(\mathscr{D}_0\downarrow F) \to \mathscr{D}_0$ is a fibration,*

(III) *$\mathrm{dom}\colon \mathscr{D}_1 \to \mathscr{D}_0$ and $\mathrm{cod}\colon \mathscr{C}_1 \to \mathscr{C}_0$ are both fibrations.*

Then for any $d \in \mathscr{D}_0$, the square

$$\begin{array}{ccc} B(F\downarrow d) & \longrightarrow & B\mathscr{C} \\ \downarrow & & \downarrow{\scriptstyle BF} \\ B(\{d\}) & \longrightarrow & B\mathscr{D} \end{array}$$

is a homotopy pullback.

Proof. We apply [ERW19, Theorem 4.9] to the commutative diagram of topological categories

$$\begin{array}{ccc} F\downarrow d & \xrightarrow{s} & \mathscr{C} \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle F} \\ \{d\} & \hookrightarrow & \mathscr{D} \end{array}.$$

The conditions of the theorem simplify greatly in this case, and we can check each individually:

1. Condition (i) holds because our categories have units and $\{d\}$ is always left fibrant (meaning the source map is a fibration); hence it suffices to assume that $\mathscr{D}$ is left fibration, which is part of assumption III.

2. Condition (ii) holds under the assumption that $\mathscr{C}$ is right fibrant (meaning the target map is a fibration), as this implies that $F\downarrow d$ is also right fibrant, and that $\mathrm{Ob}(\mathscr{D}_0\downarrow F) \to \mathscr{D}_0$ is a fibration (as $F\downarrow d \to \{d\}$ is always a fibration). This is assumptions (II) and (III).

3. Condition (iii) is assumption (I).

4. Condition (iv) is always true in this case, as $t\downarrow d$ is clearly isomorphic to $F\downarrow d$.

□

In particular, this means that $B(F\downarrow d)$ models the homotopy fiber of BF. By fibration, we mean Serre fibration, although it is possible to consider a larger class of fibrations (see [ERW19, Remark 4.12]).

Remark 3.30. As a corollary of this version of Theorem B, we obtain a slightly different version of Theorem A when each fiber $B(F\downarrow d)$ is contractible which allows us to check contractibility on fibers individually, rather than in a shrinkable way. See [ERW19, Theorem 4.7].

3.5.3 Thomason's theorem

Quillen's Theorem B gives sufficient conditions for the homotopy fiber of $BF\colon B\mathscr{C} \to B\mathscr{D}$ to be a classifying space, but it turns out that the homotopy cofiber of BF can *always* be modeled by a classifying space, with no conditions on $\mathscr{C}$, $\mathscr{D}$, or F. This result follows from a version of Thomason's homotopy colimit theorem [Tho79], which makes use of *Grothendieck constructions* on diagrams of categories. The author first learned about this application of Thomason's theorem (in the discrete case) from this stackexchange post. We first outline how the Grothendieck construction works for topological categories.

Definition 3.31. Let $D\colon I \to \mathbf{TopCat}$ be a (small) diagram of topological categories. The *Grothendieck construction* $\int_I D$ is the topological category whose objects are pairs (i, x) of $i \in I$ and $x \in D(i)_0$ and whose morphisms are $(\alpha, u)\colon (i, x) \to (j, y)$ with $\alpha\colon i \to j$ in I and $u\colon D(\alpha)(x) \to y$ in $\mathscr{D}(j)_1$. The objects are topologized as $\coprod_{i\in I} D(i)_0$ and the morphisms as a subspace of $\coprod_{\alpha\in\mathrm{Hom}(I)} D(\mathrm{cod}(\alpha))_1$.

Theorem 3.32 (Thomason)**.** *There is a natural homotopy equivalence*

$$\eta\colon \operatorname{hocolim}_I BD \xrightarrow{\simeq} B\int_I D.$$

Proof. Thomason's proof in [Tho79, §1] applies equally well to simplicial spaces as to simplicial sets, as we now detail. Similarly to the proof of Quillen's Theorem A, an essential intermediary is the topological category $\mathscr{S}(\pi)$ for the projection functor $\pi\colon \int_I D \to I$. The idea is to first define a map

$$\eta\colon \operatorname{hocolim}_I BD \simeq B(\int_I D)$$

and then show there is a zig-zag

$$\operatorname{hocolim}_I ND \xleftarrow{\lambda_1} B\mathscr{S}(\pi) \xrightarrow{\lambda_2} B(\int_I D)$$

so that $\lambda_2 \simeq \eta\circ\lambda_1$. The claim will then follow by the 2-of-3 property of equivalences, observing that both λ_1 and λ_2 are equivalences.

Consider the bisimplicial space whose (n, m)-simplices are

$$\coprod_{i_0\leftarrow\cdots\leftarrow i_n} N_m D(i_n)$$

and whose bisimplicial structure comes from the simplicial structures of N_*I and $N_*D(i)$. Observe that if we realize in the m-direction, we obtain the simplicial space from Example 2.8 that models $\operatorname{hocolim}_I BD$. Hence, by the realization lemma for bisimplicial spaces, we may define η on the diagonal of this bisimplicial space, which we do in the following way: Given $i_n \xrightarrow{\alpha_n} \dots \xrightarrow{\alpha_1} i_0$ in $N_n I$ and $c_n \xrightarrow{f_n} \dots \xrightarrow{f_1} c_0$ in $N_n D(i_n)$, we construct the length n chain of composable morphisms in $\int_I D$ whose j^{th} factor is

$$(i_j, D(\alpha_{j+1} \circ \dots \circ \alpha_n)(c_j)) \xrightarrow{(\alpha_j, D(\alpha_j \circ \dots \circ \alpha_n)(f_j))} (i_{j-1}, D(\alpha_j \circ \dots \circ \alpha_n)(c_{j-1}))$$

for $j = n, \dots, 1$. It is straightforward (albeit a bit tedious) to verify that η defines a map of simplicial spaces $\coprod_{N_* I} N_* D \to N_* \int_I D$.

Consider the topological category $\mathscr{S}(\pi)$ (as in Subsection 3.5.1) whose objects are pairs $(i \to i', (i, x \in D(i))$ and whose morphisms are commutative squares

$$\begin{array}{ccc} \pi(i, x_i) & \xrightarrow{\pi(\alpha, u)} & \pi(j, x_j) \\ \downarrow & & \downarrow \\ i' & \longleftarrow & j' \end{array} .$$

Recall from Subsection 3.5.1 that projection to second factor $\lambda_2 \colon \mathscr{S}(\pi) \to \int_I D$ induces an equivalence $B(\int_I D) \simeq B\mathscr{S}(\pi)$.

Moreover, recall that $N_* \mathscr{S}(\pi)$ is the diagonal of the bisimplicial space whose (n, m)-simplicies are

$$\coprod_{i_0 \leftarrow \dots \leftarrow i_n} N_m(\pi \downarrow i_n).$$

Realizing in the m-direction, we observe that this bisimplicial space also models $\text{hocolim}_I B(\pi \downarrow -)$. It thus suffices to show that $\text{hocolim}_I BD \simeq \text{hocolim}_I B(\pi \downarrow -)$ are equivalent in a suitable way.

For each $i \in I$, there is a continuous functor $\lambda_1^i \colon (\pi \downarrow i) \to D(i)$ which is given on objects by $(\alpha \colon i \to i', x_i \in D(i))$ to $D(\alpha)(x_i) \in D(i')$. On morphisms,

$$\begin{array}{ccc} \pi(i, x_i) & \xrightarrow{(\beta, f)} & \pi(j, x_j) \\ \downarrow{\scriptstyle \alpha_i} & \swarrow{\scriptstyle \alpha_j} & \\ i' & & \end{array} \longmapsto (D(\alpha_i)(x_i) \xrightarrow{D(\alpha_j)(f)} D(\alpha_j)(x_j)) \in D(i').$$

Note that this definition makes sense as $f \colon D(\beta)(x_i) \to x_j$ in $D(j)$, and so the continuous functor $D(\alpha_j) \colon D(j) \to D(i')$ sends f to a morphism between $D(\alpha_j)D(\beta)(x_i) = D(\alpha_i)(x_i)$ and $D(\alpha_j)(x_j)$.

For every $i \in I$, λ_1^i has a right adjoint which sends $x \in D(i)$ to $(i = i, x)$, and hence λ_1^i induces an equivalence after geometric realization. In fact, the functors λ_1^i assemble into a natural transformation $\lambda_1 \colon D \Rightarrow (\pi \downarrow -)$ of functors $I \to \mathbf{TopCat}$. Since each λ_1^i is an equivalence after taking classifying spaces, the realization lemma implies that λ_1 induces an equivalence $\text{hocolim}_I BD \simeq \text{hocolim}_I B(\pi \downarrow -)$.

Finally, we consider the composition $B\mathscr{S}(\pi) \xrightarrow{\lambda_1} \text{hocolim}_I BD \xrightarrow{\eta} B \int_I D$ and show it is homotopic to λ_2. As the nerve is fully faithful, it suffices to consider what happens on objects and morphisms. The composition $\eta \circ \lambda_1 \colon \mathscr{S}(\pi) \to \int_I D$ is the continuous functor which is given on objects by

$$(\alpha\colon i \to i', (i, x \in D(i))) \mapsto D(\alpha)(x) \in D(i') \mapsto (i', D(\alpha)(x) \in D(i')).$$

On the other hand, $\lambda_2(i \to i', x \in D(i)) = (i, x \in D(i))$. Similarly, on morphisms, $\eta \circ \lambda_1$ is given by first applying λ_1,

$$\begin{array}{ccc} \pi(i, x_i) & \xrightarrow{\pi(\beta, f)} & \pi(j, x_j) \\ {\scriptstyle \alpha_i}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_j} \\ i' & \xleftarrow[\delta]{\quad} & j' \end{array} \longmapsto (\delta\colon j' \to i'; D(\alpha_j \circ \beta)(x_i) \xrightarrow{D(\alpha_j)(f)} D(\alpha_j)(x_j) \in D(j')),$$

and then η,

$$\longmapsto (i', D(\alpha_i)(x_i)) \xrightarrow{(\delta, D(\alpha_j)(f)} (j', D(\alpha_j)(x_j))$$

where we have made use of the fact that $\alpha_i = \delta \circ \alpha_j \circ \beta$. The functor λ_2 sends a morphism as above to just $(\beta, f)\colon (i, x_i) \to (j, x_j)$. We claim there is a continuous natural transformation $H\colon \lambda_2 \Rightarrow \eta \circ \lambda_1$ which sends an object $(\alpha\colon i \to i', x_i \in D(i))$ in $\mathscr{S}(\pi)$ to the morphism in $\int_I D$ given by

$$(\alpha, \mathrm{id}_{D(\alpha)(x)})\colon (i, x \in D(i)) \to (i', D(\alpha)(x) \in D(i')).$$

Verifying that this is indeed natural and continuous, we obtain a homotopy between the two maps, as claimed. □

Definition 3.33. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor and define $C(F)$ to be the Grothendieck construction of the diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ \big\downarrow & & \\ * & & \end{array}$$

where $*$ is the discrete one-object category. That is, the objects of $C(F)$ are pairs $(x, \mathscr{I})$ where $\mathscr{I}$ is one of $\mathscr{C}, \mathscr{D}$, or $*$ and $x \in \mathrm{Ob}(\mathscr{I})$. The morphisms $(x, \mathscr{I}) \to (y, \mathscr{J})$ come in a few flavors:

- $\mathscr{I} = \mathscr{J}$, in which case it is just a morphism in $\mathscr{I}$;
- $\mathscr{I} = \mathscr{C}$ and $\mathscr{J} = *$, in which case it is the unique morphism $!_x\colon *(x) \to *$,
- $\mathscr{I} = \mathscr{C}$ and $\mathscr{J} = \mathscr{D}$, in which case it is a morphism $F(x) \to y$ in $\mathscr{D}$.

Theorem 3.34. *For any functor $F\colon \mathscr{C} \to \mathscr{D}$, there is a homtopy cofiber sequence of spaces*

$$B\mathscr{C} \xrightarrow{BF} B\mathscr{D} \to BC(F).$$

Proof. Apply Thomason's theorem to the diagram $C(F)\colon I \to \textbf{TopCat}$, where I is the span category $a \leftarrow b \to c$. In this case, the homotopy colimit is precisely the homotopy cofiber of BF. □

3.6 Some More Examples

To close out this note, we will look at some examples of topological categories and their classifying spaces, highlighting where/how the results of the previous section may be applied.

3.6.1 Topological groups

Let G be a well-based topological group, i.e. a topological group such that the inclusion of the identity is a cofibration. We saw in Example 2.6 that we can think of G as the one object category whose morphisms are the group G itself, which now has a topology. This is an example of a topological category, since the domain, codomain, and identity maps are trivially continuous and composition is continuous by assumption.

Our discussion in Subsection 3.4.1 says that if we have two groups G and H and a *continuous functor* between their associated categories which is a levelwise equivalence, then we get a homotopy equivalence on their classifying spaces (since the groups are well-based, we can apply Proposition 3.2). In particular, note that the data of such a functor is exactly the data of a continuous group homomorphism $G \to H$ which is also an equivalence.

This is related to the reduction of structure groups in bundle theory (as hinted at in Subsection 2.5.2). In particular, the statement that $G \simeq H$ is equivalent to saying that every G-bundle can have its structure group reduced to H. For example, the Gram-Schmidt algorithm gives an equivalence from $GL_n(\mathbb{R})$ to $O(n)$, and so we know that $BGL_n(\mathbb{R}) \simeq BO(n)$, and hence every real (rank n) vector bundle can have its structure group reduced from $GL_n(\mathbb{R})$ to $O(n)$. The same story can also be told for $GL_n(\mathbb{C})$ and $U(n)$.

It is important to note the differences between BG where we take the topology of G into account, and $B(G^\delta)$ where we forget the topology on G. For example, we know $B(G^\delta)$ is a $K(G^\delta, 1)$ whereas BG has the property that $\pi_i(BG) = \pi_{i-1}(G)$ (since $\Omega BG \simeq G$). So for instance $B\mathbb{R}$ is contractible, since $\mathbb{R}$ is, but $B\mathbb{R}^\delta$ is not. Typically BG and BG^δ are not homotopy equivalent, and in light of Theorem 3.23 (noting that our topological groups are most often well-based), this means their nerves are usually not Reedy fibrant.

3.6.2 Internal two-sided bar construction

We saw in Subsection 2.5.3 that we can use the two-sided bar construction to model the classifying space of a (non-topological) category. Moreover, given a functor $G\colon \mathscr{C} \to \mathbf{Top}$ which is weakly contractible (meaning there is a weak equivalence $G \to *$), then $\operatorname{hocolim} G \simeq B(*, \mathscr{C}, G) \simeq B(*, \mathscr{C}, *) = B\mathscr{C}$.

It would be helpful to have similar bar-construction models for enriched/topological categories. For enriched categories, we can bump up our definition of the two-sided simplicial bar construction to keep track of the extra topological structure by setting

$$B_n(F, \mathscr{C}, G) := \coprod_{(\operatorname{Ob}\mathscr{C})^{n+1}} F(i_0) \times \mathscr{C}(i_0, i_1) \times \cdots \times \mathscr{C}(i_{n-1}, i_n) \times G(i_n).$$

Now $B(*, \mathscr{C}, *)$ really *is* the classifying space of the topologically enriched category $\mathscr{C}$. In particular, we can use this enriched two-sided bar construction to model the classifying space of a topological group. The space $B(*, I, G)$ is an example of a so-called "weighted homotopy colimit" of G (see [RV14, Part II.]).

If $\mathscr{C}$ is actually internal, rather than just enriched, then we can define a bar construction using the language of O-graphs (see [Mey85] for the definitions). The idea is that we want

$$B_n(F, \mathscr{C}, G) = F \times_{\mathscr{C}_0} N_n(\mathscr{C}) \times_{\mathscr{C}_0} G,$$

so in particular when $F = * = G$, we get $B(*, \mathscr{C}, *) = B\mathscr{C}$. But what does it mean to take a pullback over $\mathscr{C}_0$ with functors F, G? It turns out that functors are not the right data to consider anymore, but instead we want to use the left/right "$\mathscr{C}_1$-modules over $\mathscr{C}_0$" which are associated to F and G.

That is, rather than using the functor F, we use a space $F(\mathscr{C}_0)$ along with a map $F(\mathscr{C}_0) \to \mathscr{C}_0$ and a "right action of $\mathscr{C}_1$ over $\mathscr{C}_0$" in the form of a map $F(\mathscr{C}_0) \times_{\mathscr{C}_0} \mathscr{C}_1 \to F(\mathscr{C}_0)$ (which satisfies various compatibility conditions). This action map is what gives us the face map d_0 in the bar construction. Similarly, for G, we use a space $G(\mathscr{C}_0)$ with a map $G(\mathscr{C}_0) \to \mathscr{C}_0$ and a "left action" $\mathscr{C}_1 \times_{\mathscr{C}_0} G(\mathscr{C}_0) \to G(\mathscr{C}_0)$. It then makes sense to talk about a simplicial space $B_*(F(\mathscr{C}_0), \mathscr{C}, G(\mathscr{C}_0))$ with

$$B_n(F(\mathscr{C}_0), \mathscr{C}, G(\mathscr{C}_0)) = F(\mathscr{C}_0) \times_{\mathscr{C}_0} N_n(\mathscr{C}) \times_{\mathscr{C}_0} G(\mathscr{C}_0),$$

and call its realization a two-sided bar construction.

3.6.3 The Čech complex revisited

Recall from Subsection 2.5.4 that we can form a simplicial space $C(f)$ from a map $f\colon X \to Y$ of topological spaces by defining

$$C(f)_n = X \times_Y \cdots \times_Y X$$

with $(n+1)$-factors. The ith face map omits the x_i from a tuple $(x_0, \ldots, x_n)$ and the jth degeneracy map repeats x_j. We saw in this example that if we think of $C(f)$ as just a simplicial space, then it is actually the nerve of a category $\mathscr{C}(f)$. However, because we have forgotten the topology, the classifying space of $\mathscr{C}(f)$ misses some important topological information.

But we can fix this issue with the new machinery we've developed. Specifically, we now think of $\mathscr{C}(f)$ as an internal topological category, with object space X and morphism space $X \times_Y X$, where there is a unique morphism $x_0 \to x_1$ if and only if

$(x_0, x_1) \in X \times_Y X$. For the sake of notation, we will encode the morphism $x_0 \to x_1$ as the tuple (x_0, x_1). The source and target maps are projections, the identity is the diagonal map $x \mapsto (x, x)$, and composition is given by $(x_0, x_1) \circ (x_1, x_2) \mapsto (x_0, x_2)$. The nth level of the nerve is exactly $C(f)_n$ since

$$(X \times_Y X) \times_X (X \times_Y X) \cong X \times_Y X \times_Y X.$$

Now, if we have a square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

we might ask under what conditions we have $B\mathscr{C}(f) \simeq B\mathscr{C}(f')$. From Subsection 2.5.4, we know that if both f and f' admit sections, then $B\mathscr{C}(f) \simeq Y$ and $B\mathscr{C}(f') \simeq Y'$, so the desired conclusion holds. Our discussion from Subsection 3.4.1 shows something more general: if f and f' are fibrations, then we get induced equivalences between the pullbacks $X \times_Y X \xrightarrow{\simeq} X' \times_{Y'} X'$ (because the strict pullback models the homotopy pullback in this case), and hence we have levelwise equivalences of Čech complexes. Thus $B\mathscr{C}(f) \simeq B\mathscr{C}(f')$ if the Čech complexes are Reedy cofibrant, which (per Definition 3.17) involves a fiberwise NDR pair condition on the "parameterized diagonal" map $X \to X \times_Y X$.

3.6.4 Framed flow categories

One interesting example of topological categories is related to the relatively new area of Floer homotopy theory. The idea of Floer homotopy theory is to try and associate a (stable) homotopy type to geometric information coming from a version of Floer homology. Many of the main ideas were introduced by R. Cohen, J. Jones, and G. Segal in the 1990s [CJS95a], and since then there have been many exciting developments in this field, in particular by R. Lipshitz and S. Sarkar in their work in Khovanov homology [LS14] and most recently by M. Abouzaid and A. Blumberg in their solution to the Arnold Conjecture in characteristic p [AB21]. An important concept in this work is that of a *(framed) flow category*, whose realization is meant to model a corresponding chain complex from Floer homology. There are several variants of the idea of a flow category in the literature, and sometimes the same definition will appear under a different name, or different definitions will appear under the same name. Rather than giving a precise definition, we will just highlight some of the common themes, pointing the reader to the references we have cited for specific details (see, in particular, the *compact smooth categories/Morse-Smale categories* in [Coh19, Definition 6] or the *equivariant flow categories* in [AB21, §2.1] or the *flow categories* in [LS14, §3.2]).

"Definition" 3.14. A *flow category* is a topological category $\mathscr{C}$ with a discrete space of objects and whose homspaces are compact, smooth, framed manifolds with corners. In most cases, the objects come with grading $gr\colon \mathrm{Ob}\,\mathscr{C} \to \mathbb{Z}$, and the homspaces are subject to certain conditions based on the grading, such as:

- $\mathrm{Hom}(x,x) = \{\mathrm{id}_x\}$ for all $x \in \mathrm{Ob}\,\mathscr{C}$.
- For objects $x \neq y$, if $gr(x) \leq gr(y)$ then $\mathrm{Hom}(x,y)$ is empty, and otherwise $\mathrm{Hom}(x,y)$ is $(gr(x) - gr(y) - 1)$-dimensional.
- Composition $\circ\colon \mathrm{Hom}(x,y) \times \mathrm{Hom}(y,z) \to \mathrm{Hom}(x,z)$ is an embedding into the boundary of $\mathrm{Hom}(y,z)$ in a special way. Furthermore, each point in the boundary of $\mathrm{Hom}(x,y)$ is in the image of the composition map.

These conditions can be made more precise by describing the homspaces as $\langle k\rangle$-*manifolds* (where k is the dimension of the homspace)[10]. The "framed" part of the definition comes from so-called *neat embeddings* of the homspaces into Euclidean spaces with corners.

The prototypical example of a flow category comes from Morse theory, as was also developed by Cohen, Jones, and Segal in [CJS95b]. Let $f\colon M \to \mathbb{R}$ be a Morse-Smale[11] function on a smooth, closed, finite-dimensional Riemannian manifold M, the *flow category* of f is a prototypical example of a framed flow category. The *flow category* of f is a topological category $\mathscr{C}_f$ whose objects are the discrete space of critical points and whose homspaces are the *moduli space of broken gradient flows* between them. The grading is given by the Morse index and composition is concatenation of the broken flows. The Morse-Smale condition on f ensures that the compactified moduli spaces have the extra structure necessary to make $\mathscr{C}_f$ into a framed flow category (see [Coh19] for more details).

Remark 3.35. As defined, flow categories are examples of topologically enriched categories, not internal ones. However, at least in the Morse setting, it is possible to encounter a genuine topological category if we allow *Morse-Bott functions* whose critical points may be submanifolds.

In [CJS95b], the authors show that for a Morse-Smale function $f\colon M \to \mathbb{R}$, the classifying space $B\mathscr{C}_f$ is homeomorphic to the underlying manifold M. The original paper [CJS95b] was never published, due to the fact that a few key assumptions were not in the literature at the time (cf. [Coh19, Remark on p.16]). Many of these gaps have since been addressed, and so the original proof of Cohen, Jones, and Segal is widely accepted.

In the same unpublished preprint, the authors also address the case when f is Morse but not Morse-Smale. They claim that $B\mathscr{C}_f$ is homotopy equivalent M in this case, rather than homeomorphic, and their proof uses many of the techniques

[10] Roughly, $\mathrm{Hom}(x,y)$ being a $\langle k\rangle$-manifold means that the boundary can be split up into k pieces which intersect at the corners of the manifold. For a precise definition, see [LS14, §3.1].

[11] Recall that *Morse* means the critical points of f are non-degenerate and *Morse-Smale* means that the stable and unstable manifolds of the gradient flow (formed with respect to a chosen Riemannian metric) intersect transversely.

of topological categories and simplicial spaces which we have discussed in this note. Unfortunately, there was an error in their proof (discussed in [Cal20]), although their claim is often cited and generally believed to be true. Some of the content of [Cal20] was an attempt to fix the proof, which unfortunately was not successful. At this time, the author is not aware of anything in the literature which addresses the non-Morse-Smale case, although a version of this result has been established for discrete Morse theory [NTT18].

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