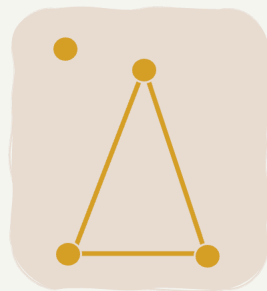


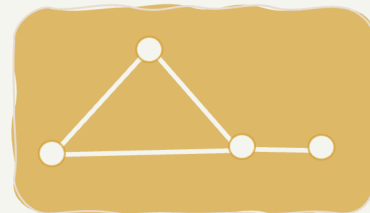
The Edge Reconstruction Conjecture

Maxine E. Calle
University of
Pennsylvania



&

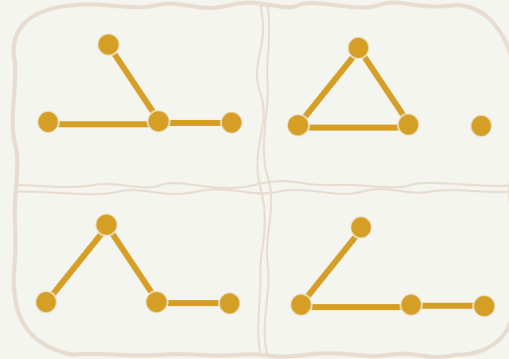
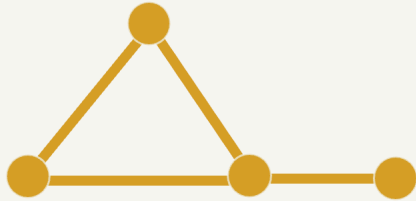
Combinatorial
K-theory



BUGCAT
October 2024

The Edge Reconstruction Conjecture

Our main objects of study are simple graphs



Defn. An edge-deleted subgraph of G is

$$G \setminus e \text{ for } e \in E(G)$$

BUGCAT

NO
loops

direction

multi-edges

infinite

The Edge Reconstruction Conjecture

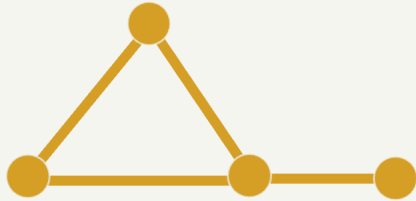
Harary (1964)
& Kelly (1942)

QUESTION

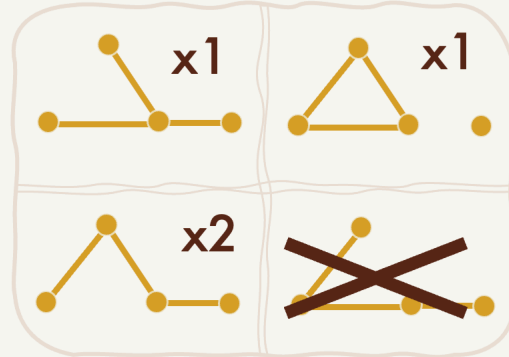
Is every graph G determined by $ED(G)$?

(up to isomorphism)

Our main objects of study are simple graphs



$$ED(\text{triangle with tail}) =$$



Defn. An edge-deleted subgraph of G is

$$G \setminus e \text{ for } e \in E(G)$$

and the edge deck of G is the multiset

$$ED(G) = \{ G \setminus e \mid e \in E(G) \}$$

BUGCAT

NO
loops

direction

multi-edges

infinite

*work one
 $n := \#$ of edges
at a time*

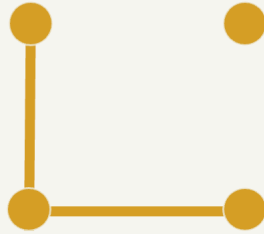
QUESTION

Is every graph G with ≥ 4 edges determined by $ED(G)$?

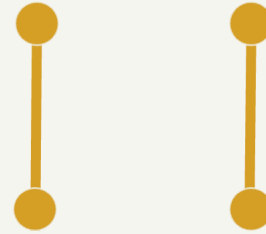
(up to isomorphism)

Note. True for $n=1$

$n=2$:

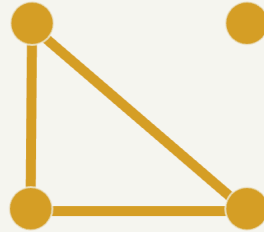


v.s.

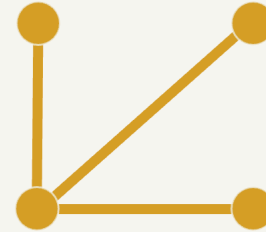


$$ED = \{ \begin{array}{c} | \\ | \\ \cdot \\ \cdot \end{array} \quad \times 2 \}$$

$n=3$:

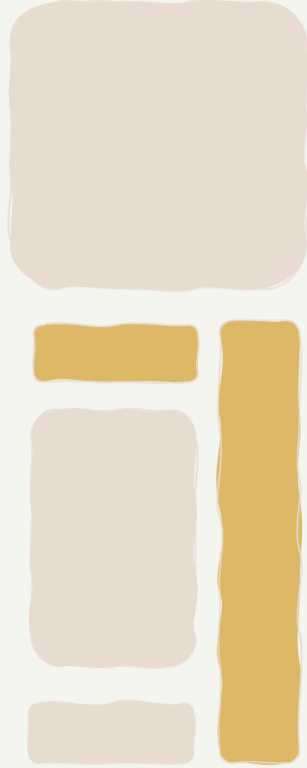
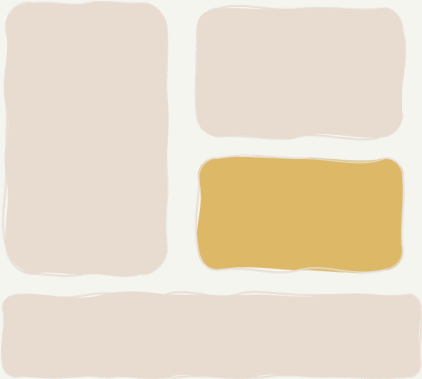


v.s.



$$ED = \{ \begin{array}{c} | \\ | \\ \cdot \\ \cdot \\ \text{---} \\ \cdot \end{array} \quad \times 3 \}$$

work one $n := \#$ of edges at a time



QUESTION

Is every graph G
with ≥ 4 edges
determined
by $ED(G)$?

(up to isomorphism)

known to be true for:

small n

regular

every vertex has
same degree

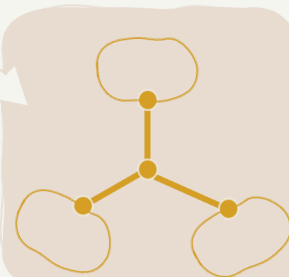
disconnected

trees
(Kelly, 1942)

complete

no induced
"claw"
subgraph
(Ellingham–Pyber–Yu,
1988)

*work one
 $n := \#$ of edges
at a time*



known to be false for:

directed
graphs

(Stockmeyer, 1977)

matroids

(Brylawski, 1975)

locally finite

every vertex has
finite degree

(Bowler–Erde–
Heinig–Lehner–
Pitz, 2017)

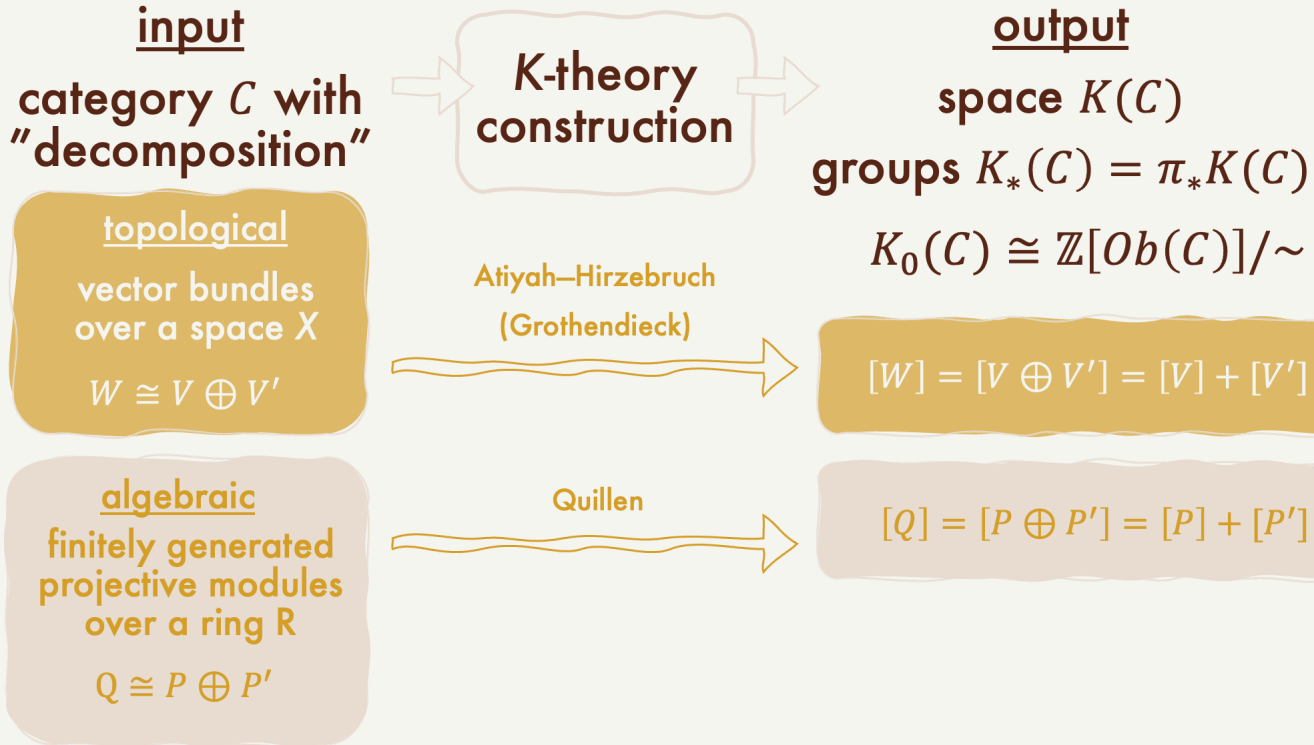
infinite

hypergraphs

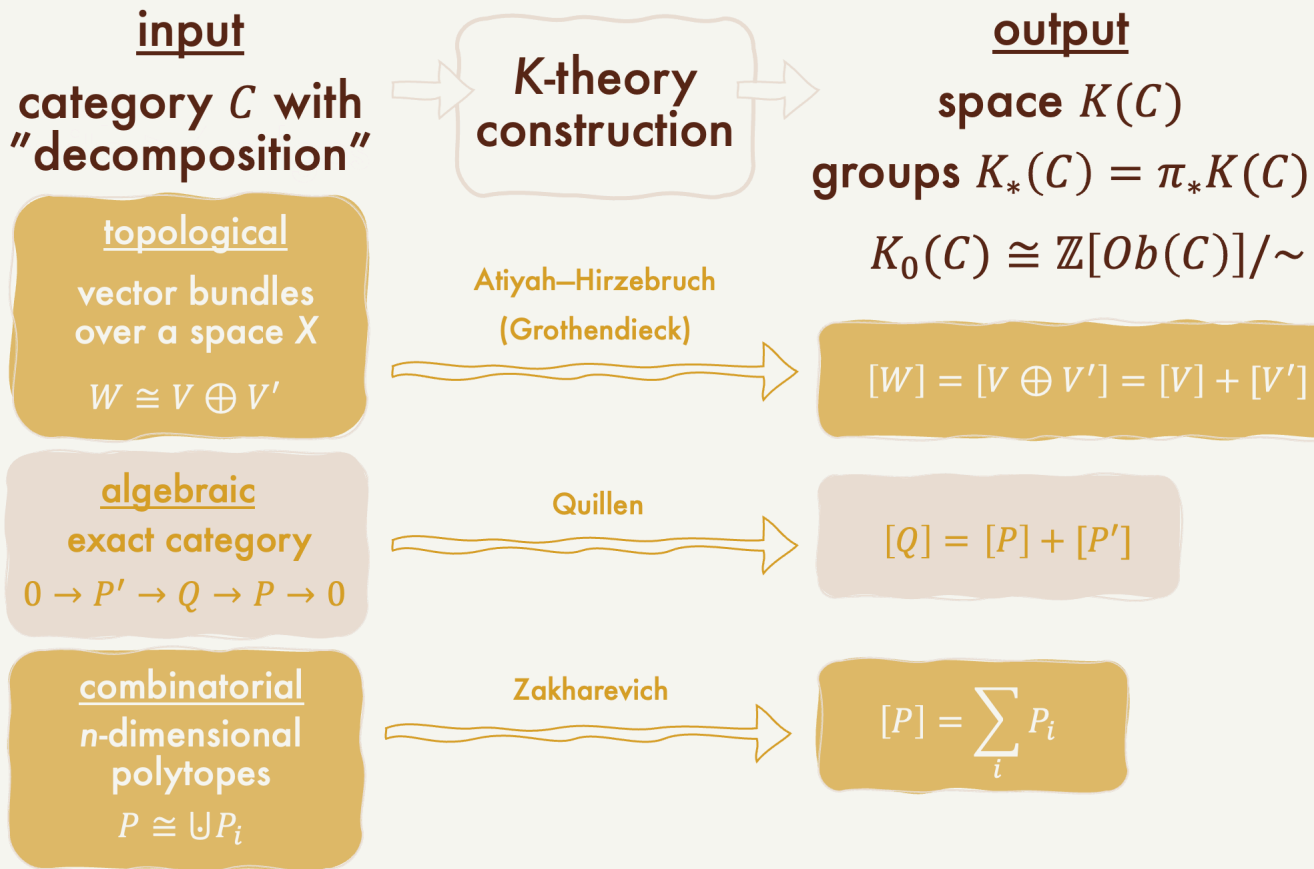
(Kocay, 1987)



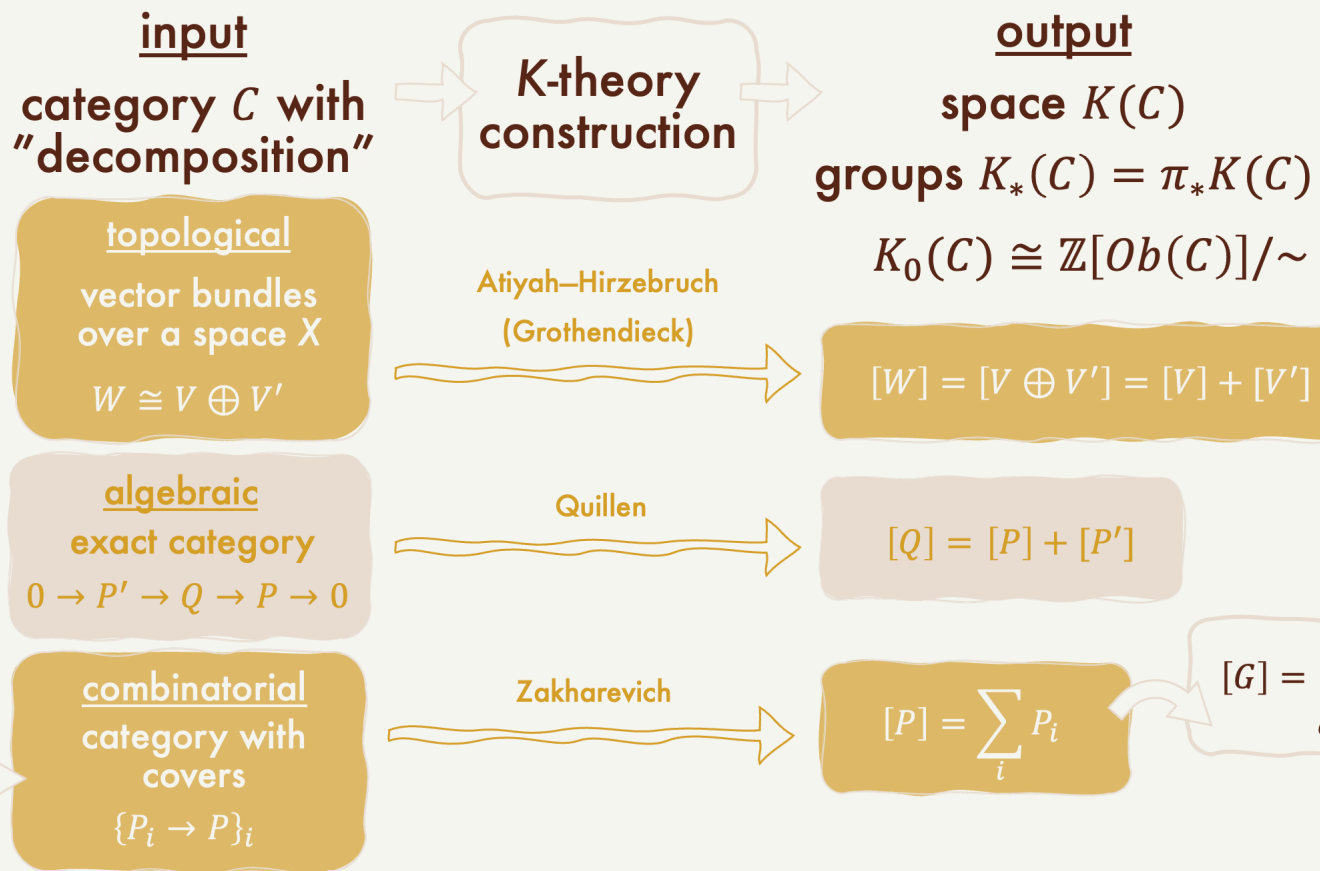
Idea. Reframe this problem in the context of K-theory



Idea. Reframe this problem in the context of K-theory



Idea. Reframe this problem in the context of K-theory



graphs with "covers" \leftrightarrow ED

Idea. Reframe this problem in the context of K-theory

graphs with
"covers" \leftrightarrow ED

combinatorial
category with
covers
 $\{P_i \rightarrow P\}_i$

Zakharevich

$$K_0(C) \cong \mathbb{Z}[Ob(C)]/\sim$$

$$[P] = \sum_i P_i$$

$$[G] = \sum_{e \in E(G)} [G \setminus e]$$

Thm. (C.-Gould) There is a category $\Gamma_{n,n-1}$ with

$$\left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of } n\text{-edge graphs} \end{array} \right\} \xrightarrow{i_n} K_0(\Gamma_{n,n-1})$$

$$\begin{array}{c} \vdots \\ \exists! \tilde{f} \\ \downarrow \\ A \end{array}$$

$$\searrow f$$

such that :

- i_n is injective iff the ERC holds for n -edge graphs
- every reconstruction invariant f factors

If $ED(G)=ED(G')$ then $f(G) = f(G')$ \leftarrow

Ex. Euler characteristic, #isolated vertices, chromatic polynomial...

Idea. Reframe this problem in the context of K-theory

graphs with "covers" \leftrightarrow ED

combinatorial category with covers
 $\{P_i \rightarrow P\}_i$

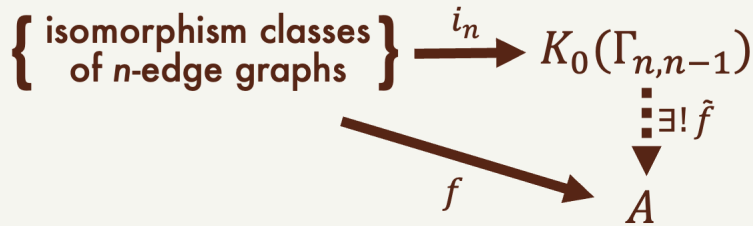
Zakharevich

$$K_0(C) \cong \mathbb{Z}[Ob(C)]/\sim$$

$$[P] = \sum_i P_i$$

$$[G] = \sum_{e \in E(G)} [G \setminus e]$$

Thm. (C.-Gould) There is a category $\Gamma_{n,n-1}$ with

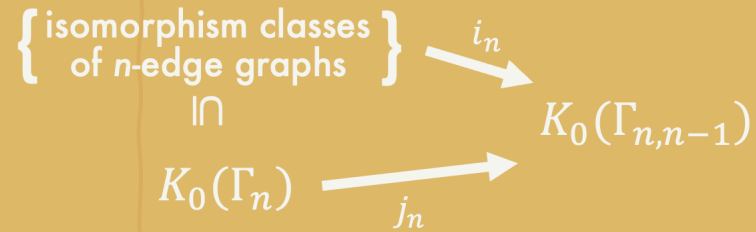


such that :

- i_n is injective iff the ERC holds for n -edge graphs
- every reconstruction invariant f factors

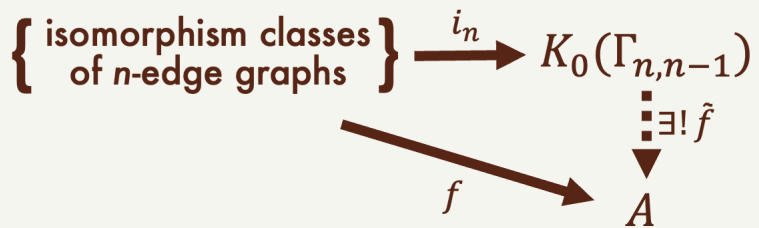
If $ED(G) = ED(G')$ then $f(G) = f(G')$ \leftarrow

Ex. Euler characteristic, #isolated vertices, chromatic polynomial...



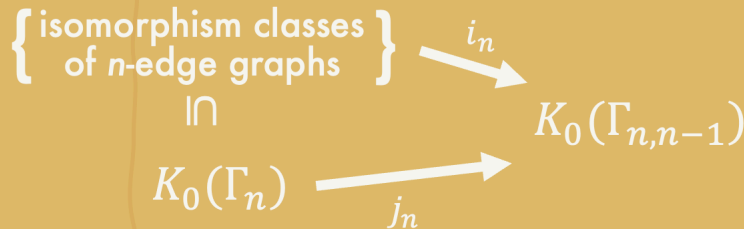
Thm. The map j_n is not injective for any n

Thm. (C.-Gould) There is a category $\Gamma_{n,n-1}$ with



such that :

- i_n is injective iff the ERC holds for n -edge graphs
- every reconstruction invariant f factors



Thm. The map j_n is not injective for any n

The Generalized Edge Reconstruction Conjecture

Is every collection G_1, \dots, G_k of n -edge graphs determined by $ED(G_1) \cup \dots \cup ED(G_k)$?

Nope!

Cor. For all n there is a minimal k_n where this fails

New Q: bounds on k_n ?

The Generalized Edge Reconstruction Conjecture

Is every collection G_1, \dots, G_k of n -edge graphs determined by $ED(G_1) \cup \dots \cup ED(G_k)$?

Nope!

Cor. For all n there is a minimal k_n where this fails

New Q: bounds on k_n ?

• $k_2 = 1$

$$\left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] - \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] = 0$$

• $k_3 = 1$

$$\left[\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right] - \left[\begin{array}{c} \bullet \\ \backslash \quad / \\ \bullet \quad \bullet \end{array} \right] = 0$$

• original ERC: $k_n > 1$ for all $n \geq 4$

• $k_4 = 2$

$$0 = \left(\left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] - \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] \right)^2$$

$$= \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] - 2 \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] + \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right]$$

• $k_5 = 2$

$$\left(\left[\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right] - \left[\begin{array}{c} \bullet \\ \backslash \quad / \\ \bullet \quad \bullet \end{array} \right] \right) \left(\left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] - \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] \right)$$

• $k_6 = 2$

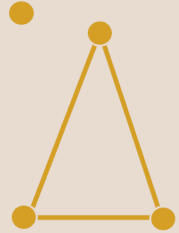
$$\left(\left[\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right] - \left[\begin{array}{c} \bullet \\ \backslash \quad / \\ \bullet \quad \bullet \end{array} \right] \right)^2$$

Thanks!

Questions?



BUGCAT
October 2024



A combinatorial K -theoretic
perspective on the Edge
Reconstruction Conjecture in
graph theory

arXiv:2402.14986

