CYCLOTOMIC SPECTRA TALK NOTES

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Abstract. The fundamental work of Dundas–Goodwillie–McCarthy relates the algebraic K-theory of a connective ring spectrum R to its topological cyclic homology TC(R), which is in turn a functor of the cyclotomic spectrum THH(R).

In this talk, the speaker will introduce a modern definition of the category of cyclotomic spectra, due to Nikolaus and Scholze. A basic example is given by THH(R) when R is a ring spectrum. The speaker will introduce several invariants of cyclotomic spectra, namely TP, TC^{-}, TC, and TR. They will also briefly mention THH with coefficients in a bimodule, and the related formalism of p-polygonic spectra. Finally, they will give the statement of the Dundas–Goodwillie–McCarthy theorem.

References: [NS18], [KN18], [HS18], [BM13], [KMN23], [Bur+23].

Idea and motivation. We want to construct counterexamples using K-theory. But to actually check the properties of these K-theory spectra we need to compute things about them. This is hard to do. Instead, the idea is to approximate K-theory with other invariants — this strategy is called *trace methods* — and use these computations to say something about the original spectra that we cared about.

In particular, the Dundas–Goodwillie–McCarthy Theorem is going to tell us that the most helpful thing we can compute is the *topological cyclic homology*, TC, and that this is a "good" approximation to K-theory in many cases of interest. But to actually construct TC, we need to first understand a different invariant called *topological Hochschild homology*, THH, and the special structure it has as a *cyclotomic spectrum*.

The goal of this talk is to introduce what cyclotomic spectra are, from the perspective of [NS18], focusing on THH as a key example. We will then discuss how to construct TC and state (but not prove) the Dundas–Goodwillie–McCarthy Theorem. Along the way, we will highlight a bunch of constructions that come from the cyclotomic structure of THH.

Notation and conventions. We will write S^1 for the circle group; this is also written as $\mathbb{T}$ in the literature. For p a prime, we write $C_p \subseteq S^1$ for the finite cyclic group of order p given by the p^{th} roots of unity. A key observation that we will make repeated use of is that $S^1/C_p \cong S^1$ as topological groups.

We will work in ∞-categories, so e.g. Sp is the ∞-category of spectra and Sp_p is the ∞-category of p-complete spectra. We will try to be consistent in writing "Map" for mapping spaces and "map" for mapping spectra.

THH **as a cyclic bar construction.** As a first step towards approximating K-theory, we can consider the *topological Hochschild homology* of a ring spectrum A.

Definition 1 (Bökstedt)**.** The *topological Hochschild homology* of A is

$$\mathrm{THH}(A) := |\ A \;\substack{\longleftarrow\\ \Longrightarrow\\ \longleftarrow}\; A \otimes A \;\substack{\longleftarrow\\ \Longrightarrow\\ \longleftarrow\\ \Longrightarrow\\ \longleftarrow}\; A \otimes A \otimes A \qquad \ldots\ |$$

where the degeneracy maps insert the unit and the face maps multiply factors, except for the last one which twisted around by a cyclic permutation and then multiplies the first two factors. This is the *cyclic bar construction* of A in spectra.

To make this construction precise in ∞-categories, one has to be a bit careful. For $\mathbb{E}_\infty$-ring spectra there is a universal property characterization due to McClure–Schwänzel–Vogt

[MSV97]. But this rough idea of the cyclic bar construction will be good enough for the purposes of this talk.

Equivariant constructions. The cyclic bar construction is the geometric realization of a *cyclic object*, which is like a simplicial object with some extra structure maps that come from an action of the cyclic group C_n on the n-simplices. For $\mathrm{THH}(A)$, this action is just given by permuting the factors. It turns out that this extra structure gives the realization an action by the circle group S^1. So, in particular, $\mathrm{THH}(A)$ has an S^1-action.

Remark 2. Everything equivariant we do in this talk will be Borel (or naïve) as opposed to genuine. While the original definition of cyclotomic spectra involved genuine equivariant structures, the Borel version will suffice for our purposes. This insight is due to Nikolaus–Scholze [NS18], basically that if what we care about is computing $\mathrm{TC}(A)$, then (at least for A bounded below) it suffices to consider the Borel equivariant structure of $\mathrm{THH}(A)$.

Once we have a spectrum with S^1-action, we can immediately get some other stuff just from general equivariant homotopy theory. In particular, we are going to care about the homotopy orbits and homotopy fixed points.

Definition 3. The category of (Borel) S^1-spectra is $\mathrm{Sp}^{BS^1} = \mathrm{Fun}(BS^1, \mathrm{Sp})$.

Recall that if G is a finite group, then a model for the ∞-category BG is the nerve of a category with a single object $*$ and a morphism for each element of G; an element $g \in G$ then "acts" on $X(*)$ via the endomorphism $X(g)$. For the compact Lie group S^1, we can model $BS^1 \simeq \mathbb{CP}^\infty$, but the intuition is still the same.

Definition 4. Given a spectrum with S^1-action $X\colon BS^1 \to \mathrm{Sp}$, its *homotopy orbit spectrum* is

$$X_{hS^1} = \mathrm{colim}_{BS^1} X$$

and its *homotopy fixed point spectrum* is

$$X^{hS^1} = \lim_{BS^1} X.$$

The two constructions above are like the "homotopy coherent" version of take orbits and fixed points, respectively. There is one more important construction we can do, called the *Tate construction*, which is like fixed points modulo a kind of "norm."

The idea of the Tate construction comes from Tate cohomology: if G is a finite group and M a G-module, then there is a norm map from the G-orbits of M to the G-fixed points given by

$$\begin{aligned} \mathrm{Nm}\colon M_G &\to M^G \\ [m] &\mapsto \sum_{g \in G} g \cdot m. \end{aligned}$$

This map crucially allows us to define long exact sequences between group homology and group cohomology of G with coefficients in M, which gives us Tate cohomology $\hat{H}^*(G; M)$.

We want to do the same kind of thing for G-spectra; if G is a finite group, there is a norm map $\mathrm{Nm}\colon X_{hG} \to X^{hG}$ and the *Tate construction* is the cofiber

$$X^{tG} := \mathrm{cofib}(X_{hG} \xrightarrow{\mathrm{Nm}} X^{hG}) \in \mathrm{Sp}.$$

For S^1, we need to suspend first — there is a norm map $\Sigma X_{hS^1} \to X^{hS^1}$ and the Tate construction is defined to be

$$X^{tS^1} := \mathrm{cofib}(\Sigma X_{hS^1} \xrightarrow{\mathrm{Nm}} X^{hS^1}) \in \mathrm{Sp}.$$

Note that since X^{tS^1} is the cofiber, there is a canonical map $\mathrm{can}\colon X^{hS^1} \to X^{tS^1}$. This construction has some nice properties:

(1) The functor $(-)^{tG}\colon \mathrm{Sp}^{BS^1} \to \mathrm{Sp}$ is an exact functor of stable ∞-categories

(2) $(-)^{tS^1}$ is a lax symmetric monoidal functor such that $(-)^{hS^1} \xrightarrow{can} (-)^{tS^1}$ is also lax symmetric monoidal.

For THH(A), these constructions get some special names.

Definition 5. The *negative cyclic homology* of A is the homotopy fixed point spectrum

$$\mathrm{TC}^-(A) := \mathrm{THH}(A)^{hS^1}.$$

The *topological periodic homology* of A is the Tate construction

$$\mathrm{TP}(A) := \mathrm{THH}(A)^{tS^1}.$$

Note that there is the canonical map

$$\mathrm{can}\colon \mathrm{TC}^-(A) \to \mathrm{TP}(A).$$

Remark 6. We will often want to work one prime p at a time, instead considering the p-completions $\mathrm{THH}(A)^\wedge_p$, $\mathrm{TC}^-(A)^\wedge_p$ and $\mathrm{TP}(A)^\wedge_p$, as well as the p-completion of the canonical map.

Categories of cyclotomic spectra. If X is a spectrum with S^1-action, we can always restrict to an action of $C_p \leq S^1$ for any fixed prime p. Now the homotopy orbits and homotopy fixed points have a residual $S^1/C_p \cong S^1$-action, and there is a similar norm map $X_{hC_p} \to X^{hC_p}$ but *now in S^1-spectra*. Thus when we take the Tate construction

$$X^{tC_p} := \mathrm{cofib}(X_{hC_p} \to X^{hC_p})$$

for this C_p-action, X^{tC_p} inherits a residual action by $S^1/C_p \cong S^1$.

A cyclotomic spectrum is a S^1-spectrum X with some extra structure: S^1-equivariant *Frobenius maps* $\phi\colon X \to X^{tC_p}$ for each prime p. We can also work one prime at a time.

Definition 7. A **p-cyclotomic spectrum** is $X \in \mathrm{Sp}^{BS^1}$ with an S^1-equivariant map $\phi_p : X \to X^{tC_p}$ called the *Frobenius*. A morphism of p-cyclotomic spectra is a map of S^1-spectra $f\colon X \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{\phi_X} & X^{tC_p} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{tC_p}} \\ Y & \xrightarrow[\phi_Y]{} & Y^{tC_p} \end{array}.$$

Remark 8. Our definition differs slightly from others in the literature, where a p-cyclotomic spectrum may be required to have an action by C_{p^∞} rather than S^1. We note that C_{p^∞} arises as the "p-power torsion" subgroup within S^1; in particular, $(BS^1)^\wedge_p \simeq (BC_{p^\infty})^\wedge_p$, so in the p-complete setting there is no meaningful difference.

The ∞-category of p-cyclotomic spectra (or p-typical cyclotomic spectra) CycSp_p is constructed as a *lax equalizer* of the functors

$$\mathrm{Sp}^{BS^1} \underset{(-)^{tC_p}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Sp}^{BS^1}.$$

In general, a lax equalizer of two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is computed as a pullback

$$\begin{array}{ccc} \mathrm{LEq}(F,G) & \longrightarrow & \mathcal{D}^{\Delta^1} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathcal{C} & \xrightarrow[(F,G)]{} & \mathcal{D} \times \mathcal{D} \end{array};$$

note that an object is $(c, Fc \xrightarrow{f} Gc)$ and a 1-morphism is the data of a 1-morphism in $\mathcal{C}$ that makes the relevant square commute.

One may similarly define the ∞-category of cyclotomic spectra CycSp where the maps must be compatible with the Frobenii across all primes p, i.e. the lax equalizer of

$$\mathrm{Sp}^{BS^1} \underset{\prod_p (-)^{tC_p}}{\overset{\prod_p \mathrm{id}}{\rightrightarrows}} \prod_p \mathrm{Sp}^{BS^1}.$$

Defining these ∞-categories as lax equalizers immediately implies some nice properties:

(1) Equivalences of p-cyclotomic spectra are given by underlying equivalences.
(2) CycSp_p is a stable ∞-category.
(3) The forgetful functor $\mathrm{CycSp}_p \to \mathrm{Sp}^{BS^1}$ is exact and preserves all (small) colimits.
(4) For all primes p, there is a restriction map $\mathrm{CycSp} \to \mathrm{CycSp}_p$.
(5) CycSp_p inherits a symmetric monoidal structure from Sp.

THH **as a cyclotomic spectrum.** We want to construct a Frobenius map $\mathrm{THH}(A) \to (\mathrm{THH}(A))^{tC_p}$. To do so, we use something called the *Tate diagonal*.

First, observe that we have a natural way to map any spectrum X into a spectrum with C_p-action by taking the diagonal $X \to X^{\otimes p}$, where C_p permutes the factors of $X^{\otimes p}$. However, $(-)^{\otimes p}$ is not an exact functor (it is not even additive), but post-composing with the Tate construction fixes this issue.

Theorem 9. *For every $X \in \mathrm{Sp}$, there is a map $X \xrightarrow{\Delta_p} (X^{\otimes p})^{tC_p}$ called the Tate diagonal which extends to a natural transformation*

$$\mathrm{Sp} \underset{((-)^{\otimes p})^{tC_p}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Sp} \quad (\Downarrow \Delta_p).$$

Moreover, there is a unique such natural transformation Δ_p which is also symmetric monoidal.

The Tate diagonal has the nice property that it takes $\mathbb{E}_n$-algebras to $\mathbb{E}_n$-algebra maps. Moreover, if X is bounded below, then Δ_p is a p-completion. We can use the Tate diagonal to construct Frobenius maps. Before examining $\mathrm{THH}(A)$, we first discuss an easier example.

Example 10. The sphere spectrum $\mathbb{S}$ with the trivial S^1-action is a p-cyclotomic spectrum. The S^1-equivariant Frobenius map $\phi_p \colon \mathbb{S} \to \mathbb{S}^{tC_p}$ can be constructed using the Tate diagonal:

$$\phi_p \colon \mathbb{S} \xrightarrow{\Delta_p} (\mathbb{S}^{\otimes p})^{tC_p} \xrightarrow{\cong} \mathbb{S}^{tC_p} \simeq \mathbb{S}_p^\wedge.$$

More generally, note that this composition factors as

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\Delta_p} & (\mathbb{S}^{\otimes p})^{tC_p} \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle =} \\ \mathrm{map}(BC_p, \mathbb{S}) \simeq \mathbb{S}^{hC_p} & \xrightarrow[\mathrm{can}]{} & \mathbb{S}^{tC_p} \end{array}$$

where $p \colon BC_p \to *$ is the collapse map. This composition can be made equivariant, using the fact that $\mathbb{S}$ has trivial action and recalling various adjunctions involving mapping into/out of objects with trivial action. In particular, p^* and the equivalence $\mathrm{Map}(BC_p, \mathbb{S}) \simeq \mathbb{S}^{hC_p}$ are S^1-equivariant, so ϕ_p is as well (since can is equivariant by construction).

We now sketch how to use the Tate diagonal to construct the Frobenius for THH; the full proof is a bit involved, see [NS18, §III.2].

Proposition 11. *There is a Frobenius map $\phi_p \colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$.*

Sketch. The idea is to apply the Tate diagonal to the cyclic bar construction and use some formal properties. Consider the cyclic bar construction on A and apply the Tate diagonal at each level:

$$(A^{\otimes p})^{tC_p} \leftrightarrows ((A \otimes A)^{\otimes p})^{tC_p} \quad ((A \otimes A \otimes A)^{\otimes p})^{tC_p} \qquad \cdots$$

The Tate diagonal gives a map of cyclic objects $\Delta_p\colon \mathrm{THH}(A) \to \mathrm{colim}_n((A^{\otimes np})^{tC_p})$. One then argues that there is a map $\mathrm{colim}_k((A^{\otimes kp})^{tC_p}) \to (\mathrm{colim}_k(A^{\otimes kp}))^{tC_p} \simeq \mathrm{THH}(A)^{tC_p}$. □

These Frobenii, across all p, give $\mathrm{THH}(A)$ the structure of a cyclotomic spectrum.

Topological cyclic homology. We can finally define $\mathrm{TC}(A)$, making use of a previous example that $\mathbb{S}$ with trivial action can be extended to a p-cyclotomic spectrum via the Tate diagonal.

Definition 12. The *topological cyclic homology* of A is

$$\mathrm{TC}(A) := \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}, \mathrm{THH}(A)).$$

If A is connective (so in particular $\mathrm{THH}(A)$ is bounded below), then there is an equivalence

$$\mathrm{TC}(A)^\wedge_p \simeq \mathrm{map}_{\mathrm{CycSp}_p}(\mathbb{S}, \mathrm{THH}(A))^\wedge_p$$

and there is an additional description of $\mathrm{TC}(A)^\wedge_p$ as an equalizer. We need the connective assumption as this allows us to make use of the Tate orbit lemma in the identification below.

Theorem 13. *There is an equivalence*

$$\mathrm{TC}(A)^\wedge_p \simeq \mathrm{eq}\left(\mathrm{TC}^-(A)^\wedge_p \underset{\phi}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{TP}(A)^\wedge_p \right).$$

By a slight abuse of notation, the map can in the diagram above is the p-completion of the canonical map $\mathrm{THH}(A)^{hS^1} \to \mathrm{THH}(A)^{tS^1}$. The map ϕ is a bit more complicated: first, since the Frobenius map ϕ_p is S^1-equivariant, we may consider

$$(\phi_p)^{hS^1}\colon \mathrm{THH}(A)^{hS^1} \to (\mathrm{THH}(A)^{tC_p})^{hS^1} \simeq (\mathrm{THH}(A)^{tC_p})^{h(S^1/C_p)},$$

where the equivalence is merely using our favorite fact $S^1/C_p \cong S^1$. Now, the Tate orbit lemma implies that we may miraculously "cancel fractions" and obtain

$$(\mathrm{THH}(A)^{tC_p})^{h(S^1/C_p)} \simeq \mathrm{THH}(A)^{tS^1}.$$

The p-completion of the composition

$$\mathrm{THH}(A)^{hS^1} \xrightarrow{(\phi_p)^{hS^1}} (\mathrm{THH}(A)^{tC_p})^{hS^1} \simeq (\mathrm{THH}(A)^{tC_p})^{h(S^1/C_p)} \simeq \mathrm{THH}(A)^{tS^1}$$

is the map ϕ in the equalizer above.

Remark 14. Historically, (integral) $\mathrm{TC}(A)$ was constructed via a fracture square involving the p-typical versions of topological cyclic homology, defined to be the fiber term in the sequence:

$$\mathrm{TC}(A,p) \to \mathrm{TR}(A,p) \xrightarrow{F-1} \mathrm{TR}(A,p).$$

Here $\mathrm{TR}(A,p) = \lim_n \mathrm{THH}(A)^{C_{p^n}}$ is p-typical *topological restriction homology*, which has a certain "Frobenius" endomorphism F. Even though the modern perspective on TC does not explicitly make use of TR, TR can still be useful to detect certain properties of cyclotomic spectra; I've been told this will be discussed in some of the talks on Tuesday.

The Dundas–Goodwillie–McCarthy Theorem. These constructions are supposed to be approximations to K-theory, and so far we haven't said anything about that. Part of the motivation for Bökstedt's construction of $\mathrm{THH}(A)$ was to make use of the *Dennis trace* $K(A) \to \mathrm{THH}(A)$ to detect p-torsion in $K_*(\mathbb{Z})$. It turns out that the Dennis trace factors

$$K(A) \to \mathrm{TC}(A) \to \mathrm{TC}^-(A) \to \mathrm{THH}(A)$$

where $K(A) \to \mathrm{TC}(A)$ is the *cyclotomic trace*. We can view each of $\mathrm{THH}(A)$, $\mathrm{TC}^-(A)$, and $\mathrm{TC}(A)$ as better and better approximations to $K(A)$. The Dundas–Goodwillie–McCarthy theorem makes the statement "TC is a good approximation to K-theory" more precise.

Theorem 15. *If $A \to B$ is a nilpotent extension*[1] *of connective $\mathbb{E}_1$-rings, then there is a pullback square*

$$\begin{array}{ccc} K(A) & \longrightarrow & \mathrm{TC}(A) \\ \downarrow & \lrcorner & \downarrow \\ K(B) & \longrightarrow & \mathrm{TC}(B) \end{array}.$$

Remark 16. An equivalent formulation of this theorem is that

$$K^{\mathrm{inv}} := \mathrm{fib}(K \to \mathrm{TC})$$

is constant along nilpotent extensions. One important example of a nilpotent extension is $A \to H\pi_0 A$ for any connective $\mathbb{E}_1$-ring A.

We will end this talk by highlighting a construction that the proof of this theorem makes use of, although we will not have time to detail the proof itself.

Definition 17. Let A be a ring and M be a (A, A)-bimodule. Define

$$\mathrm{THH}(A, M) := \left|\ M \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; M \otimes A \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; M \otimes A \otimes A \qquad \cdots \right|$$

where the face and degeneracy maps are like those in the cyclic bar construction, but make use of the bimodule structure on M to multiply on the left or right. This is the *topological Hochschild homology of A with coefficients in M*.

Observe that if $M = A$, then this recovers the cyclic bar construction of A. In general, the simplicial object above is not cyclic and hence the realization may not have an S^1-action. However, it has a structure similar to that of a p-cyclotomic spectrum, in that there is a kind of Frobenius map

$$\mathrm{THH}(A, M) \to \mathrm{THH}^{(p)}(A, M)^{tC_p}$$

for a "p-fold" version of THH with coefficients.[2] Similar to the usual Frobenius map on THH, the construction of this map makes use of the Tate diagonal. This structure is not quite p-cyclotomic, but is what is called *p-polygonic*, which is a slightly weaker notion. In particular, every p-cyclotomic spectrum determines a p-polygonic one by forgetting the S^1-equivariance to just C_p-equivariance.

Remark 18. This p-polygonic structure plays a crucial technical role in [Bur+23]. I've been told that this idea will reappear in some of the talks on Thursday on cyclotomic constancy.

[1] This means the map is surjective on π_0 with nilpotent kernel.

[2] In [Bur+23], this construction is denoted with a subscript pentagon.

References

[BM13] Thomas G. Goodwillie Bjørn Ian Dundas and Randy McCarthy. "The local structure of algebraic K-theory". In: *Algebra and Applications* 18 (2013).

[Bur+23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank. *K-theoretic counterexamples to Ravenel's telescope conjecture*. 2023. arXiv: 2310.17459 [math.AT]. URL: https://arxiv.org/abs/2310.17459.

[HS18] Lars Hesselholt and Peter Scholze. "Arbeitsgemeinschaft: Topological Cyclic Homology". In: *Oberwolfach Rep.* 15.2 (2018), pp. 805–940.

[KMN23] Achim Krause, Jonas McCandless, and Thomas Nikolaus. *Polygonic spectra and TR with coefficients*. 2023. arXiv: 2302.07686 [math.AT]. URL: https://arxiv.org/abs/2302.07686.

[KN18] Achim Krause and Thomas Nikolaus. *Lectures on Topological Hochschild Homology and cyclotomic spectra*. 2018. URL: https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Lectures.pdf.

[MSV97] J. McClure, R. Schwänzl, and R. Vogt. "$\mathrm{THH}(R) \cong R \otimes S^1$ for $\mathbb{E}_\infty$ ring spectra". In: *J. Pure Appl. Algebra* 121.2 (1997), pp. 137–159.

[NS18] Thomas Nikolaus and Peter Scholze. "On topological cyclic homology". In: *Acta Math.* 221.2 (2018), pp. 203–409.