Notes on Waldhausen's higher algebraic K-theory

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These notes came together mostly in preparation for my Ph.D. qualifying exam on higher algebraic K-theory in 2022. There are likely typos, errors, and other mistakes throughout. Read at your own peril!

Contents

1 Introduction

Higher algebraic K-theory provides a framework for extracting invariants from algebraic and geometric objects by producing a space $K(\mathscr{C})$ from a suitable category $\mathscr{C}$ which encodes the objects of interest in some way. Beginning with Grothendieck's definition of K_0 for exact and abelian categories, the subject developed rapidly through Quillen's introduction of higher K-groups, which revealed deep connections with number theory, algebraic geometry, and topology. Waldhausen's approach to algebraic K-theory arose from the need to extend these ideas beyond purely algebraic settings. Many naturally occurring categories in topology, such as categories of spaces or spectra, do not admit exact sequences in the classical sense, yet they possess notions of cofibration and weak equivalence that govern how objects are built and compared up to homotopy. Waldhausen categories abstract these features, providing a flexible framework in which one can define and study K-theory and apply it to a wide range of algebraic and homotopical contexts.

These notes provide an introduction to Waldhausen's approach to higher algebraic K-theory. Section 2 introduces Waldhausen categories and their associated Grothendieck group K_0, and we discuss some standard examples (such as exact/abelian categories and retractive spaces). In Section 3, we describe Waldhausen's definition of higher algebraic K-theory via the $S_\bullet$-construction, and we explain how the $S_\bullet$-construction compares with Quillen's Q-construction in situations where both are defined. Section 4 focuses on the fundamental theorems that aid computations in higher algebraic K-theory, particularly the additivity theorem.

In addition to Waldhausen's paper [Wal85], standard references for this material include [Rog10, Sri93, Wei13]. There are other approaches to higher algebraic K-theory which are beyond the scope of these notes, particularly Segal's K-theory of symmetric monoidal categories [Seg74] and the K-theory of stable ∞-categories [BGT13].

2 Waldhausen categories

A Waldhausen category comes with two distinguished classes of morphisms: cofibrations and weak equivalences. These classes of morphisms interact "how we think they should" based on familiar examples of categories with a notion of cofibration and weak equivalence. For example, if $\mathscr{C}$ is a model category and $\mathscr{C}_c$ is its full subcategory of cofibrant objects, then $\mathscr{C}_c$ forms a Waldhausen category with the inherited cofibrations and weak equivalences from $\mathscr{C}$.

Waldhausen categories are also *pointed*, meaning $\mathscr{C}$ comes with a distinguished zero object 0. This assumption ensures that if we turn $\mathscr{C}$ into a space via some functorial process, the resulting space will come with a natural basepoint from this zero object.

Definition 2.1. A *category with cofibrations* is a pointed category $\mathscr{C}$ with a wide subcategory *co*$\mathscr{C}$ of *cofibrations* (indicated with $\rightarrowtail$ arrows) satisfying the following requirements:

co(i) the isomorphisms of $\mathscr{C}$ are in *co*$\mathscr{C}$,

co(ii) for every object $X \in \mathscr{C}$, the arrow $0 \to X$ is a cofibration, co(iii) if $X \rightarrowtail Y$ is a cofibration and $X \to Z$ is any morphism, then we have the following pushout diagram:

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, tail] & Y \cup_X Z \end{tikzcd} .$$

These axioms imply that we can do the following constructions in $\mathscr{C}$:

- (Coproducts) For any two objects $X, Y \in \mathscr{C}$, the coproduct $X \coprod Y$ exists as the pushout $X \cup_0 Y$.
- (Cofibration sequences) For any cofibration $i \colon X \rightarrowtail Y$ in $\mathscr{C}$, the pushout $Y \cup_X 0$ (of i along $X \to 0$) is called the *quotient* (or *cofiber*) of i and is written Y/X. We call $X \rightarrowtail Y \twoheadrightarrow Y/X$ a cofibration sequence and refer to the canonical map $Y \twoheadrightarrow Y/X$ as a *quotient map.* Note that Y/X is not unique, but is defined only up to canonical isomorphism.

Almost all of our constructions will deal with categories with cofibrations, but we will add in weak equivalences at the end to get Waldhausen categories.

Definition 2.2. A *Waldhausen category* is a category with cofibrations $\mathscr{C}$ with a subcategory of *weak equivalences $w\mathscr{C}$* (indicated with $\xrightarrow{\sim}$ arrows) such that

w(i) the isomorphisms of $\mathscr{C}$ are in $w\mathscr{C}$,

w(ii) gluing axiom: for every commutative diagram

$$\begin{tikzcd} Y \arrow[d, "\sim"'] & Z \arrow[l] \arrow[r, tail] \arrow[d, "\sim"] & X \arrow[d, "\sim"] \\ Y' & Z' \arrow[l] \arrow[r, tail] & X' \end{tikzcd} ,$$

the induced map $X \cup_Z Y \to X' \cup_{Z'} Y'$ is a weak equivalence.

Although technically a Waldhausen category is a tuple $(\mathscr{C}, co\mathscr{C})$, we will often just write $\mathscr{C}$. We say that a Waldhausen category is *saturated* if whenever $g \circ f$ is a weak equivalence, then f is a weak equivalence if and only if g is.

For example, any category with cofibrations becomes a Waldhausen category by setting $w\mathscr{C}$ to be $iso\mathscr{C}$ or all of $\mathscr{C}$. A functor between Waldhausen categories must preserve all the relevant structure: zero objects, cofibrations, weak equivalences, and pushouts along cofibrations; such functors are called *exact functors.* We let **Wald** denote the category of Waldhausen categories and exact functors between them.

2.1 K_0 of a Waldhausen category

The idea of the K_0-group of a Waldhausen category is to decompose objects according to how they fit into cofibration sequences.

Definition 2.3. The 0^{th} K-group of a Waldhausen category $\mathscr{C}$ is the free Abelian group $K_0(\mathscr{C})$ on $X \in \operatorname{Ob}\mathscr{C}$ subject to the relations

1. if $X \xrightarrow{\sim} Y$, then $[X] = [Y]$;
2. if $A \rightarrowtail X \twoheadrightarrow X/A$ then $[X] = [A] + [X/A]$.

The second relation implies that $[0] = 0$, $[X \coprod Y] = [X] + [Y]$, and $[X \cup_Z Y] = [X] + [Y] - [Z]$; the first relation then implies that $[X] = 0$ whenever $X \simeq 0$.

Remark 2.4. In order to avoid set-theory problems, we need the collection of objects of $\mathscr{C}$ to be set-sized (more accurately, the collection of weak equivalence classes of objects needs to be set-sized). However, we can extend the theory a little bit from small categories to *essentially small* categories; a Waldhausen category $\mathscr{C}$ is essentially small if there is an exact functor $F\colon \mathscr{C} \to \mathscr{D}$ which is an equivalence of categories for some small Waldhausen category $\mathscr{D}$. In this case, we set $K_0(\mathscr{D}) = K_0(\mathscr{C})$.

Any exact functor between Walhausen categories will induce a homomorphism on their 0^{th} K-groups since all the relevant structure is preserved. Thus we get a functor $K_0\colon$ **Wald** $\to$ **Ab** from Waldhausen categories and exact functors to Abelian groups. We will see in later sections that this extends to a spectrum (or generalized cohomology theory if you prefer).

One important feature of K_0 is that it "splits" sequences of exact functors.

Theorem 2.5 (K_0 additivity)**.** *Suppose that $F, F', F''\colon \mathscr{C} \to \mathscr{D}$ are exact functors between Waldhausen categories with natural transformations $F' \Rightarrow F \Rightarrow F''$ so that*

$$F'X \rightarrowtail FX \twoheadrightarrow F''X$$

for all $X \in \mathscr{C}$. Then $[FX] = [F'X] + [F''X]$ in $K_0(\mathscr{D})$, so $K_0(F) = K_0(F') + K_0(F'')$ as maps $K_0(\mathscr{C}) \to K_0(\mathscr{D})$.

This theorem is the 0^{th} level of Waldhausen's additivity theorem, which we will discuss further in Section 4, and it can be extremely useful for computations in K-theory. Induction on the number of exact functors in the sequence yields a generalized Euler characteristic.

Another helpful theorem is the Localization Theorem, which applies in the situation when a category with cofibrations comes with two notions of weak equivalence. Let $\mathscr{C}$ be a category with cofibrations and two classes of weak equivalences, $v\mathscr{C} \subseteq w\mathscr{C}$, both of which make $\mathscr{C}$ into a Waldhausen category. Denote these Waldhausen category structures by $\mathscr{C}_v$ and $\mathscr{C}_w$, respectively. Let $\mathscr{C}^w$ denote the full subcategory of w-acyclic objects in $\mathscr{C}_v$, i.e. those objects X such that $0 \rightarrowtail X$ is a weak equivalence in $\mathscr{C}_w$. Note that the inclusions $\mathscr{C}^w \to \mathscr{C}_v \to \mathscr{C}_w$ are all exact functors.

Theorem 2.6 (K_0 localization)**.** *Suppose that $w\mathscr{C}$ is saturated and that every map $X \to Y$ in $\mathscr{C}$ factors as*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, tail] & & Y \\ & Z \arrow[ur, "\sim_v"'] & \end{tikzcd}$$

for $Z \xrightarrow{\sim_v} Y$ in $v\mathscr{C}$. Then the inclusion functors induce an exact sequence of Abelian groups

$$K_0(\mathscr{C}^w) \to K_0(\mathscr{C}_v) \to K_0(\mathscr{C}_w) \to 0.$$

Proof sketch. The map $K_0(\mathscr{C}_v) \to K_0(\mathscr{C}_w)$ is surjective because $v\mathscr{C} \subseteq w\mathscr{C}$, so we are introducing more relations of type (1) in K_0. The composition of the two maps is 0 by the definition of $\mathscr{C}^w$. So we just need to show that $K_0(\mathscr{C}_w)$ is the cokernel of $K_0(\mathscr{C}^w) \to K_0(\mathscr{C}_v)$. Let L denote the cokernel, so by its universal property we get a map $L \to K_0(\mathscr{C}_w)$. We claim that sending $[C]$ to $[C]$ induces the inverse map $K_0(\mathscr{C}_w) \to L$. $\square$

2.2 Examples

We now examine a few important examples of Waldhausen categories and their 0^{th} K-groups.

2.2.1 Abelian categories and exact categories

Abelian categories generalize things like Abelian groups and chain complexes, or other settings in which we can do homological algebra. An important example is $\mathbf{P}(R)$, the category of projective modules over a ring R. In general, we want to be able to add objects and morphisms and have things like kernels and cokernels with the right universal properties.

Definition 2.7. A category $\mathscr{C}$ is *additive* if it is **Ab**-enriched (meaning each homset has an Abelian group structure such that composition is a morphism in **Ab**, i.e. bilinear, and contains all finite “biproducts” (product = coproduct) which are written $\oplus$.

Definition 2.8. A category $\mathscr{C}$ is *Abelian* if it is additive and (i) every morphism has a kernel and a cokernel (ii) every monic arrow is the kernel of some morphism and every epi arrow is the cokernel of some morphsim.[1]

In an Abelian category $\mathscr{C}$, a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is *exact* if $\ker(g)$ is equal to $\operatorname{im}(f) := \ker(Y \to \operatorname{coker}(f))$. Per usual, a longer sequence is called *exact* if it is exact at each spot, and *short exact sequence* refers to an exact sequence of the form

$$0 \to X \to Y \to Z \to 0.$$

The type of functor between Abelian categories that we want to consider is called an *exact functor*, which is a functor $F\colon \mathscr{C} \to \mathscr{D}$ so that the maps $\mathscr{C}(X, Y) \to \mathscr{D}(FX, FY)$ are group homomorphisms (i.e. F is an **Ab**-enriched functor). An additive functor is said to be *exact* if it preserves exact sequences, meaning for every exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{C}$, we have an exact sequence $0 \to FX \to FY \to FZ \to 0$ in $\mathscr{D}$.

An exact category is an additive subcategory of an Abelian one, the idea being that we can talk about exact sequences without necessarily knowing that all the kernels and cokernels exist.

Definition 2.9. A *exact* category $\mathscr{C}$ is an additive category with a collection of sequences $\mathcal{E}$ in $\mathscr{C}$ of the form

$$0 \to X \xrightarrow{i} Y \xrightarrow{j} Z \to 0$$

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[1] Recall that an arrow $f\colon X \to Y$ is *monic* if it is “left-cancellable” ($fe_1 = fe_2$ implies $e_1 = e_2$ for $e_1, e_2\colon \hat{X} \to X$) and *epi* if it is “right-cancellable” ($g_1 f = g_2 f$ implies $g_1 = g_2$ for $g_1, g_2\colon Y \to \hat{Y}$).

which satisfies the following conditions: there is an Abelian category $\mathscr{A}$ which contains $\mathscr{C}$ as a full (embedded) subcategory such that (i) E is the class of all sequences in $\mathscr{C}$ which are exact in $\mathscr{A}$, (ii) $\mathscr{C}$ is *closed under extensions* in $\mathscr{A}$ in the sense that if we have an exact sequence $0 \to X \xrightarrow{i} Y \xrightarrow{j} Z \to 0$ in $\mathscr{A}$ with $X, Z \in \mathscr{C}$ then Y is isomorphic to an object in $\mathscr{C}$.

Sometimes we may also need the extra condition of $\mathscr{C}$ being *closed under surjections*, which means whenever $f \colon X \to Y$ is a surjection in $\mathscr{C}$ then $\ker(f) \in \mathscr{C}$. Note that an Abelian category $\mathscr{A}$ is always naturally an exact category (which is closed under surjections) by just taking $\mathcal{E}$ to be all the exact sequences in $\mathscr{A}$. We think of exact categories as living inside of Abelian ones, and we've just "forgotten" about some of the kernels and cokernels. For this reason, many of our definitions for exact categories come directly from analogous definitions for Abelian categories. For example, a functor between exact categories is called *exact* and it is required to preserve short exact sequences. If the inclusion of a subcategory into an exact (Abelian) category is exact, then the subcategory is called an *exact (Abelian) subcategory.*

Remark 2.10. Every exact category is naturally a Waldhausen category by taking the cofibrations to be the admissible monics and the weak equivalences to be the isomorphisms.

Since the only weak equivalences are isomorphism, $K_0(\mathscr{C})$ of an exact category is just the free Abelian group on objects $X \in \operatorname{Ob}\mathscr{C}$ with a relation $[Y] = [X] + [Z]$ for every short exact sequence $0 \to X \to Y \to Z \to 0$.

If we give $\mathscr{C}$ an exact structure by declaring $\mathcal{E}$ to be only the "split" short exact sequences of the form $0 \to X \to X \oplus Y \to Y \to 0$, then the two definitions coincide. We see that an Abelian category comes with two exact structures: the "natural" one coming from all short exact sequences and the "split" one coming from just the split short exact sequences, and the resulting K_0 groups are different. We will always consider the former structure unless we specify otherwise.

Example 2.11. Let R be a ring and let $\mathbf{P}_{\text{fg}}(R)$ denote the category of finitely generated projective R-modules. Then $K_0(\mathbf{P}_{\text{fg}}(R))$ is just the usual Grothendieck group of R; that is, the group completion of the monoid $(iso\mathbf{P}_{\text{fg}}(R), \oplus)$, i.e. isomorphism classes of finitely generated projective modules under direct sum, subject to the relation $[P \oplus Q] = [P] + [Q]$.

Remark 2.12. Why do we need to impose the finitely generated condition? If we allowed any projective module, then K_0 would just be 0. This is because of something called the *Eilenberg swindle.* Suppose P is a projective R-module, so there is some Q and n such that $P \oplus Q \cong R^n$. Consider the projective modules

$$\begin{aligned} R^\infty &\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \dots \\ &\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \dots \\ &\cong P \oplus R^\infty. \end{aligned}$$

Then, in K_0, we would have $[P] = 0$. In general, we typically need to impose some sort of finiteness condition on the objects of our Waldhausen categories to avoid the Eilenberg swindle.

Example 2.13. If $\mathscr{C}$ is an exact category, so is the category of chain complexes $\mathbf{Ch}(\mathscr{C})$, by declaring the exact sequences to be levelwise. We can also consider bounded chain complexes $\mathbf{Ch}^b(\mathscr{C})$. It turns out that $K_0(\mathbf{Ch}(\mathscr{C})) = 0$ (Eilenberg swindle) but $K_0(\mathbf{Ch}^b(\mathscr{C})) \cong K_0(\mathscr{C})$.

Chain complexes provide some of the more historically important examples of Waldhausen categories. Cofibrations are levelwise-monics and the weak equivalences are the quasi-isomorphisms. Let $\mathscr{C}$ be an exact category and let $\mathbf{Ch}(\mathscr{C})$ denote chain complexes in $\mathscr{C}$. The Eilenberg swindle shows that $K_0(\mathbf{Ch}(\mathscr{C})) = 0$ (by using the chain complex which is the identity on some object every third morphism or so). However if we restrict our attention to *bounded* chain complexes $\mathbf{Ch}^b(\mathscr{C})$ in $\mathscr{C}$, then $K_0(\mathbf{Ch}^b(\mathscr{C})) \cong K_0(\mathscr{C})$.

2.2.2 Retractive spaces

Let $\mathbf{CW}$ be the category of based CW complexes and cellular maps between them, and let $\mathbf{CW}_f$ be the subcategory of finite pointed CW complexes. We get a Waldhausen structure by taking the cofibrations to be injective maps and weak equivalences to be weak homotopy equivalences. Then $K_0(\mathbf{CW}) = 0$ because of the Eilenberg swindle but $K_0(\mathbf{CW}_f) \cong \mathbb{Z}$. This is because you can use the pushout relation to show that $K_0(\mathbf{CW})$ is generated by $[S^0]$ and the reduced Euler characteristic is a surjective map onto $\mathbb{Z}$.

Example 2.14. We detail how this goes for spheres; the case of more general finite CW complexes is similar. Recall that S^n can be constructed as a pushout

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\
D^n \arrow[r] & S^n
\end{tikzcd}$$

by gluing the northern and southern hemispheres (disks) along the S^{n-1} equator. This implies that $[S^n] = 2[D^n] - [S^{n-1}]$ in $K_0(\mathbf{CW}_f)$. Since $D^n \simeq *$, we thus have $[S^n] = -[S^{n-1}]$ and we can iterate this argument to obtain $[S^n] = (-1)^n[S^0]$.

In fact, the reduced Euler characteristic is the *universal* additive invariant in the following sense: suppose that F is any rule that assigns a finite pointed CW complex to some integer such that

- F is a homotopy invariant: if $X \simeq Y$ then $F(X) = F(Y)$,
- F is additive: if $A \to X \to X/A$ is a cofiber sequence of pointed finite CW complexes, then $F(X) = F(A) + F(X/A)$,
- $F(*) = 0$,
- $F(S^0) = 1$,

then F is in fact the reduced Euler characteristic. More generally, if F satisfies the first three conditions then it is completely determined by its value on S^0. In fact, the first three conditions are equivalent to F factoring through $K_0(\mathbf{CW}_f)$, and this observation leads to the following result.

Proposition 2.1. *Let A be an Abelian group. Then any group homomorphism $F\colon K_0(\mathbf{CW}_f) \to A$ is completely determined by the value of $F(S^0)$. The reduced Euler characteristic is the unique map $K_0(\mathbf{CW}_f) \to \mathbb{Z}$ which sends S^0 to 1.*

More generally, let X be a CW complex and let $\mathcal{R}(X)$ be the category of CW complexes Y obtained from X by attaching cells and having a retraction $r\colon Y \to X$. Let $\mathcal{R}_f(X)$ denote the subcategory of those Y which are obtained by attaching only finitely many cells, and let $\mathcal{R}_{fd}(X)$ be the subcategory of Y which are homotopy finite (i.e. retracts up to homotopy of spaces in $\mathcal{R}_f(X)$). This is a generalization of the previous example, where $X = *$. The K-theory of the category $R(X)$ is often called the *A-theory* of X, written $A(X)$, so we could instead write $K_0(\mathcal{R}_{hf}(X)) = A_0(X)$.

2.2.3 The extension category

Let $\mathscr{C}$ be a Waldhausen category. Its *extension category*, $\mathcal{E}(\mathscr{C})$, is another Waldhausen category whose objects are cofibration sequences $X \rightarrowtail X' \twoheadrightarrow Y$. A morphism from $X \rightarrowtail Y \twoheadrightarrow Z$ to $X' \rightarrowtail Y' \twoheadrightarrow Z'$ is a natural transformation of diagrams, which is to say it consists of maps (letter)→(letter'). A cofibration in $\mathcal{E}(\mathscr{C})$ is one where $X \to X'$, $Z \to Z'$, and $X' \cup_X Y \to Y'$ are. Weak equivalences are just given componentwise.

Proposition 2.2. *The map which sends $X \rightarrowtail Y \twoheadrightarrow Z$ to (X, Z) induces an isomorphism $K_0(\mathcal{E}(\mathscr{C})) \to K_0(\mathscr{C}) \times K_0(\mathscr{C})$.*

Proof. There is a section $K_0(\mathscr{C}) \times K_0(\mathscr{C}) \to K_0(\mathcal{E}(\mathscr{C}))$ which sends (X, Z) to $X \rightarrowtail X \vee Z \twoheadrightarrow Z$. The claim follows by noting that $[Y] = [X] + [Z] = [X \vee Z]$ whenever $X \rightarrowtail Y \twoheadrightarrow Z$ is a cofibration sequence. ☐

The extension category and its generalizations are important for defining the $S_\bullet$-construction. We can generalize the idea of $\mathcal{E}(\mathscr{C})$ and consider the category of n-filtrations $0 \rightarrowtail A_1 \rightarrowtail \ldots \rightarrowtail A_n$. Morphisms are natural transformations of diagrams. Weak equivalences are defined levelwise, and a morphism is a cofibration if for all $1 \leq i < j < k \leq n$, the map of cofibration sequences

$$\begin{tikzcd}
A_{ji} \arrow[r, tail] \arrow[d] & A_{ki} \arrow[r, two heads] \arrow[d] & A_{kj} \arrow[d] \\
A'_{ji} \arrow[r, tail] & A'_{ki} \arrow[r, two heads] & A'_{kj}
\end{tikzcd}$$

is a cofibration in $\mathcal{E}(\mathscr{C})$. We claim that $K_0(\mathcal{E}_n) = K_0(\mathscr{C})^{\oplus n}$; to see this, iterate the additivity theorem.

3 Higher algebraic K-theory

Waldhausen's higher algebraic K-theory is built for Waldhausen categories and uses bisimplicial machinery [Wal85]. The main thing to understand is the $S_\bullet$-construction, which is a simplicial category that encodes sequences of cofibrations in $\mathscr{C}$. The K-theory space of $\mathscr{C}$ is (loops on) the realization of this simplicial category. In fact, $K(\mathscr{C})$ is an ∞-loop space (and hence a connective spectrum), whose deloopings are given by iterating the $S_\bullet$-construction.

Exact categories exist naturally as Waldhausen categories, and so we can also compare Waldhausen's construction to Quillen's Q-construction [Qui73, Gra76]. As we might expect, these two definitions of K-theory coincide, which we discuss in Subsection 3.3.2. I read (on the internet though, so who knows how true this is) that Quillen was aware of Waldhausen's $S_\bullet$-construction when he developed the Q-construction, but preferred the Q-construction because it built K-theory from the nerve of a category.

One of Waldhausen's motivations for his K-theory theory construction was to be able to take the algebraic K-theory of spaces, encoded in the category of retractive spaces $R(X)$ from Subsection 2.2.2. The resulting spectrum is called the *A-theory of X* and is closely related to the K-theory of the group ring $K(\mathbb{Z}[\pi_1(X)])$, which encodes geometric information about X. We won't discuss these ideas further here, but encourage the reader who likes manifolds to see e.g. [Wal85, Wal82, Wal87, WJR13].

3.1 The $S_\bullet$-construction

The $S_\bullet$-construction turns a category with cofibrations $\mathscr{C}$ into a simplicial category, i.e. for each $n \geq 0$ we get a category $S_n\mathscr{C}$ and these categories come with face and degeneracy functors between them. Roughly, $S_n\mathscr{C}$ is a category consisting of diagrams of cofibration sequences in $\mathscr{C}$. Once we have the simplicial category $S_\bullet\mathscr{C}$, we can freely manipulate it using all the tools of simplicial homotopy theory, e.g. taking its nerve, realizing it, etc. Through these manipulations we will end up with a space whose homotopy groups will be the algebraic K-theory of $\mathscr{C}$. The miraculous fact is that this definition will actually generalize Quillen's definitions for exact and Abelian categories.

Recall that for each $n \geq 0$, the poset category $[n]$ has objects $\{0, \dots, n\}$ and morphisms $i \to j$ whenever $i \leq j$. We can consider the arrow category of $[n]$, which has objects equal to the morphisms of $[n]$ and a morphism $i \to j$ to $i' \to j'$ in $ar[n]$ is a commutative square

$$\begin{tikzcd}
i \arrow[r] \arrow[d] & i' \arrow[d] \\
j \arrow[r] & j'
\end{tikzcd}.$$

Note that whenever $i \leq j \leq k$, we have a morphism from $i \to j$ to $j \to k$ in $ar[n]$, as well as $i \to k$ to $j \to k$. These particular morphisms will be important in the construction of the category $S_n\mathscr{C}$.

Definition 3.1. Let $\mathscr{C}$ be a category with cofibrations. The category $S_n\mathscr{C}$ will be a subcategory of the functor category $Fun(ar[n], \mathscr{C})$ (i.e. $ar[n]$-shaped diagrams in $\mathscr{C}$). For a functor $A\colon ar[n] \to \mathscr{C}$, write $A_{ij} := A(i \to j)$ for each each morphism $i \to j$. The category $S_n\mathscr{C}$ consists of functors A subject to the following conditions:

1. $A_{ii} = 0$ for all i,

2. for all $i \leq j \leq k$, the induced morphism $A_{ij} \rightarrowtail A_{ik}$ is a cofibration and fits into the cofibration sequence $A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk}$.

Morphisms are natural transformations between these functors.

Let's explore what happens for small n. When $n = 0$, $ar[0]$ is just the trivial category and by requirement (1) we see that $S_0\mathscr{C}$ is just the trivial subcategory 0. When $n = 1$, we have one object $0 \to 1$ and no non-identity morphisms and so an object of $S_1\mathscr{C}$ just picks out an object A_{01} of $\mathscr{C}$. Requirement (2) just says that $0 \rightarrowtail A_{01}$ (which is just axiom co(ii) for a category with cofibrations). A natural transformation is just a morphism in $\mathscr{C}$, so we see that $S_1\mathscr{C}$ is $\mathscr{C}$ itself. When $n = 2$, an object of $S_2\mathscr{C}$ looks like a diagram

$$\begin{tikzcd}
0 = A_{00} \arrow[r, tail] & A_{01} \arrow[r, tail] \arrow[d, two heads] & A_{02} \arrow[d, two heads] \\
& 0 = A_{11} \arrow[r, tail] & A_{12} \arrow[d, two heads] \\
& & 0 = A_{22}
\end{tikzcd},$$

i.e. a cofibration $A_{01} \rightarrowtail A_{02}$ along with a choice of quotient A_{12}. The way we've written the diagram above generalizes for larger n, where the horizontal arrows are always cofibrations and the vertical arrows are always quotients.

$$\begin{tikzcd}
0 = A_{00} \arrow[r, tail] & A_{01} \arrow[r, tail] \arrow[d, two heads] & \dots \arrow[r, tail] & A_{0n} \arrow[d, two heads] \\
& 0 = A_{11} \arrow[r, tail] & \dots \arrow[r, tail] & A_{1n} \arrow[d, two heads] \\
& & & \vdots \arrow[d, two heads] \\
& & & 0 = A_{nn}
\end{tikzcd}.$$

We get cofibration sequences by going right any number of steps and then down any number of steps. For example, both $A_{01} \rightarrowtail A_{02} \twoheadrightarrow A_{12}$ and $A_{01} \rightarrowtail A_{0n} \twoheadrightarrow A_{(n-1)n}$ are cofibration sequences.

We can reinterpret objects of $S_n\mathscr{C}$ as n-filtrations on $\mathscr{C}$. A n-filtration on an object $A \in \mathscr{C}$ is a sequence of n cofibrations $A_0 \rightarrowtail A_1 \rightarrowtail \dots \rightarrowtail A_n = A$. To satisfy requirement (1), we will assume our filtrations start at the zero object, with $A_0 = 0$. To get from an element of $S_n\mathscr{C}$ to an n-filtration, we just forget everything except the first row. To get from an n-filtration to an element of $S_n\mathscr{C}$, we need to *choose* quotients $A_{ij} := A_j/A_i$ (remember quotients are only defined up to natural isomorphism) and then we can fill out the rest of the diagram by the universal property of pushouts and the axiom co(iii). Both requirements (1) and (2) are satisfied basically by definition; note that the second part of requirement (2) is like a version of one of the isomorphism theorems in abstract algebra: $(A_k/A_i)/(A_j/A_i) \cong A_k/A_j$.

Remark 3.2. We'd like to say that $S_n\mathscr{C}$ is equivalent to the category $F_n\mathscr{C}$ of n-filtrations on $\mathscr{C}$, but this isn't quite right (at least when we try to build the face and degeneracy maps, as we'll see in a second), since we need to keep track of our choices of quotients. Waldhausen's approach [Wal85, §1] is to keep track of these choices in the data of the category of filtrations itself, which he denotes $F_n^+\mathscr{C}$. Then we do indeed have $S_n\mathscr{C} \simeq F_n^+\mathscr{C}$, and so we can think of our simplicial category as $F_\bullet^+\mathscr{C}$ if we prefer. Of course, to know that $S_\bullet\mathscr{C}$ (equivalently $F_\bullet^+\mathscr{C}$) is a simplicial category, so we need to say what the face and degeneracy maps are. These structure maps will look very much like the structure maps for a nerve of a category; we will first define them on $F_\bullet^+\mathscr{C}$ and then reinterpret them for $S_\bullet\mathscr{C}$.

Definition 3.3. For $0 \leq j \leq n$, the j^{th} *degeneracy map* $s_j\colon S_n\mathscr{C} \to S_{n+1}\mathscr{C}$ inserts the identity $A_j \rightarrowtail A_j$ in the j^{th} spot (as a map $F_n^+\mathscr{C} \to F_{n+1}^+\mathscr{C}$). On an element of $S_n\mathscr{C}$, this duplicates the j^{th} column, inserting identities horizontally, and inserts a row of $0's$ as the j^{th} row (the quotients A_j/A_j).

For $0 \leq i \leq n$, we have the i^{th} face map $d_i\colon S_n\mathscr{C} \to S_{n-1}\mathscr{C}$. For $0 < i < n$, these maps just replace two cofibrations $A_{i-1} \rightarrowtail A_i \rightarrowtail A_{i+1}$ with their composition $A_{i-1} \rightarrowtail A_{i+1}$, which (as an element of $S_n\mathscr{C}$) removes the i^{th} column by composing horizontally and also removes the i^{th} row (where we've quotiented by A_i) by composing vertically. When $i = n$, the face map d_n removes the last column by just ignoring $A_{n-1} \rightarrowtail A_n$ (note that this also removes the last row which is just $a_{nn} = 0$). Dually, for d_0, we just remove the first row (which also removes the first column $A_{00} = 0$).

Note that when we view the structure maps on $F_\bullet^+\mathscr{C}$, the face map d_0 is the only structure map which requires us to make a choice of quotients. That is, the other s_j and d_i ($i \neq 0$) maps can be defined on $F_n^+\mathscr{C}$ by just saying "omit this arrow" or "add in this arrow," but for d_0 we cannot simply omit the arrow $0 = A_0 \rightarrowtail A_1$, since we have assumed all our cofibration sequences start at 0 (in order to satisfy requirement (1)). What we need d_0 to do to a cofibration sequence $0 = A_0 \rightarrowtail A_1 \rightarrowtail \ldots \rightarrowtail A_n$ is omit A_0 *and* make functorial choices of quotients A_j/A_1 for all $1 < j \leq n$. This is the benefit of working with $S_n\mathscr{C}$: its construction is more clearly functorial with respect to a maps in $\mathbf{\Delta}$. But it will be helpful to keep both of these formulations in mind and move back and forth between them freely.

Now, it is straightforward (albeit tedious) to check that these structure maps satisfy the simplicial identities, and so $S_\bullet\mathscr{C}$ is indeed a simplicial category. (In fact, $S_\bullet\mathscr{C}$ is a simplicial Waldhausen category whenever $\mathscr{C}$ is Waldhausen, meaning each $S_n\mathscr{C}$ is Waldhausen.) Our ultimate goal is to turn this into a space, somehow, which we do with the machinery of bisimplicial sets. Since each $S_n\mathscr{C}$ is a category, we are free to take its nerve $NS_n\mathscr{C}$, and it is perhaps no surprise that we can assemble all of this data into a bisimplicial set

$$N_*S_\bullet\mathscr{C}\colon \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{Set}$$

by mapping $([m],[n]) \mapsto N_mS_n\mathscr{C}$. The (m,n)-bisimplices of $|S_\bullet w\mathscr{C}|$ look like chains of m-composable weak equivalences of length n cofibration sequences (thinking of $S_\bullet w\mathscr{C}$ as

$F_*^+\mathscr{C}$).

$$\begin{tikzcd}
0 \arrow[r, tail] & A_1^0 \arrow[r, tail] \arrow[d, "\sim"] & \dots \arrow[r, tail] & A_n^0 \arrow[d, "\sim"] \\
0 \arrow[r, tail] & A_1^1 \arrow[r, tail] \arrow[d, "\sim"] & \dots \arrow[r, tail] & A_n^1 \arrow[d, "\sim\ ."] \\
 & \vdots \arrow[d, "\sim"] & \ddots & \vdots \arrow[d, "\sim"] \\
0 \arrow[r, tail] & A_1^m \arrow[r, tail] & \dots \arrow[r, tail] & A_n^m
\end{tikzcd}$$

Recall that we can turn a bisimplicial set into a space in a few different ways, but they all end up being homeomorphic. By realizing in just one component we get simplicial sets $[n] \mapsto BS_n\mathscr{C}$ and $[m] \mapsto |N_m S_\bullet \mathscr{C}|$. The realizations of these two simplicial sets are actually homeomorphic and are also homeomorphic to the realization of the diagonal $[n] \mapsto N_n S_n \mathscr{C}$. This space will give us the higher K-groups of a Waldhausen category, as we shall see in the next section.

3.2 Waldhausen's higher K-theory

Let $\mathscr{C}$ be a Waldhausen category and consider the subcategory $w\mathscr{C}$ of weak equivalences. Let $N_*^w S_n \mathscr{C} \subset N_* S_n \mathscr{C}$ be the sub-simplicial set of the nerve comprised of morphisms of diagrams that are pointwise valued in $w\mathscr{C}$. Let $wS_\bullet\mathscr{C}$ be the bisimplicial set $N_*^w S_\bullet \mathscr{C}$. The homotopy groups of the realization of this bisimplicial set are the K-groups of $\mathscr{C}$, up to a shift.

Definition 3.4. The *algebraic K-theory space* of a Waldhausen category $\mathscr{C}$ is $K(\mathscr{C}) := \Omega|wS_\bullet\mathscr{C}|$. The *algebraic K-theory groups of $\mathscr{C}$* are

$$K_i(\mathscr{C}) := \pi_i(\Omega|wS_\bullet\mathscr{C}|) = \pi_{i+1}(|wS_\bullet\mathscr{C}|).$$

Since everything involved in constructing $K(\mathscr{C})$ is functorial, an exact functor $F\colon \mathscr{C} \to \mathscr{D}$ will induce a map $KF\colon K(\mathscr{C}) \to K(\mathscr{D})$. Hence $K\colon \mathbf{Wald} \to \mathbf{Top}_*$ and $K_i := \pi_i K\colon \mathbf{Wald} \to \mathbf{Ab}$ are functors. It turns out that K is actually valued in H-spaces.

Proposition 3.1. *The coproduct $\coprod\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ extends to each category $wS_n\mathscr{C}$ and induces an H-space structure on $K(\mathscr{C})$. Moreover, an exact functor $F\colon \mathscr{C} \to \mathscr{D}$ induces a map of H-spaces $KF\colon K(\mathscr{C}) \to K(\mathscr{D})$.*

We have claimed that the functor K_i lands in Abelian groups, which is clear for $i \geq 0$ (since $\pi_i K(\mathscr{C}) = \pi_{i+1}(|wS_\bullet\mathscr{C}|)$ is Abelian for $i \geq 1$). For $i = 0$, we will show that our "new" definition coincides with the old (Abelian) one from Subsection 2.1. We will show this after a quick lemma.

Lemma 3.5. *Let $\mathscr{C}$ be a Waldhausen category. Then $|S_\bullet w\mathscr{C}|$ is a connected CW complex and its (right) 1-skeleton is $\Sigma Bw\mathscr{C}$.*

Proof. Recall that there are multiple ways to realize a bisimplicial set, but the resulting spaces are homeomorphic. We'll work with the realization of the right simplicial space $[n] \mapsto BS_n w\mathscr{C}$ which we know is CW because it is homeomorphic to the realization of the diagonal simplicial set.[2] There is only one 0-cell in this CW complex because $BS_0 w\mathscr{C} = B(*) = *$ (remembering that $S_0 w\mathscr{C}$ is always the trivial category). Thus all higher cells must have this one point as a vertex, so the CW complex is connected.

To see that the (right) 1-skeleton is $\Sigma BS_1 w\mathscr{C}$, we first observe that the 1-cells will have both ends attached to this point $*$. Note that $S_1 w\mathscr{C} \cong w\mathscr{C}$, so $BS_1 w\mathscr{C} \cong Bw\mathscr{C}$. The right 1-skeleton can be written as the pushout

$$\begin{tikzcd}
Bw\mathscr{C} \times S^0 \arrow[r] \arrow[d, hook] & * \arrow[d] \\
Bw\mathscr{C} \times D^1 \arrow[r] & {|S_\bullet w\mathscr{C}|^{(1)}}
\end{tikzcd} .$$

Note that $Bw\mathscr{C}$ is based at 0. This pushout (taken in based spaces) is exactly the reduced suspension $\Sigma w\mathscr{C}$. $\square$

The adjoint of the inclusion $\Sigma Bw\mathscr{C} \to |S_\bullet w\mathscr{C}|$ is a map $Bw\mathscr{C} \to \Omega|S_\bullet w\mathscr{C}|$. In particular this gives us maps $\mathrm{Ob}\,\mathscr{C} \to \pi_1(|S_\bullet w\mathscr{C}|)$ and $\mathrm{Hom}(w\mathscr{C}) \to \pi_2(|S_\bullet w\mathscr{C}|)$.

Proposition 3.2. *If $\mathscr{C}$ is a Waldhausen category then $\pi_1(|S_\bullet w\mathscr{C}|)$ is isomorphic to $K_0(\mathscr{C})$ in the sense of Subsection 2.1.*

Proof. We will show that $\pi_1(|S_\bullet w\mathscr{C}|)$ has the same generators and relations as $K_0(\mathscr{C})$, by computing $|S_\bullet w\mathscr{C}|$ as the realization of the simplicial space $[n] \mapsto BS_n w\mathscr{C}$. By simplicial set stuff, we know $\pi_1(|S_\bullet w\mathscr{C}|)$ is the free group on $\pi_0(BS_1 w\mathscr{C}) \cong \pi_0(Bw\mathscr{C})$ modulo the relations $d_1(x) = d_2(x)d_0(x)$ for all $x \in \pi_0(BS_2 w\mathscr{C})$. But $\pi_0(Bw\mathscr{C})$ is precisely the weak equivalence classes of objects in $\mathscr{C}$, so we have a generator $[X]$ for every $X \in \mathrm{Ob}\,\mathscr{C}$ with the relation $[X] = [Y]$ whenever there is a weak equivalence between X and Y. Moreover, $S_2 w\mathscr{C}$ is the category whose objects are cofibrations $X \rightarrowtail Y$ and the relation $d_1(X \rightarrowtail Y) = d_2(X \rightarrowtail Y)d_0(X \rightarrowtail Y)$ says precisely that $[X] = [Y] + [X/Y]$. $\square$

Remark 3.6. There are natural homotopy equivalences $|wS_\bullet\mathscr{C}| \simeq \Omega|wS_\bullet S_\bullet C\mathscr{C}|$ (cf. [Wei13, Lemma 8.5.4 and Remark 8.5.5]) and so $K(\mathscr{C}) \simeq \Omega^2|wS_\bullet S_\bullet\mathscr{C}|$. In fact, $K(\mathscr{C})$ is an infinite loop space, which we can see by iterating the $S_\bullet$-construction, forming multisimplicial Waldhausen categories $S^n_\bullet = S_\bullet S_\bullet \dots S_\bullet\mathscr{C}$. The space $|wS^n_\bullet\mathscr{C}|$ is the loop space of $|wS^{n+1}_\bullet\mathscr{C}|$ and so the sequence of spaces

$$K(\mathscr{C}) = \Omega|wS_\bullet\mathscr{C}|, |wS_\bullet\mathscr{C}|, |wS^2_\bullet\mathscr{C}|, \dots, |wS^n_\bullet\mathscr{C}|, \dots$$

forms a connective Ω-spectrum $K\mathscr{C}$ called the *K-theory spectrum* of $\mathscr{C}$. We'll return to this idea after discussing the additivity theorem in Section 4.

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[2] This is something I was a bit worried about at first. We'd like to say that $|[n] \mapsto BS_n w\mathscr{C}|$ is CW with n-skeleton coming from $BS_n w\mathscr{C}$. We can get a CW structure on $|[n] \mapsto BS_n w\mathscr{C}|$ via the homeomorphism with $|diag(N_* S_\bullet w\mathscr{C})|$, but we might worry that this isn't the CW structure that we want. But, if we check the proofs around identifications of realizations of bisimplicial sets (e.g. in [Rog10, §6.5]), we see that the homeomorphism gives us exactly the CW structure we want. So everything is okay!

3.3 Comparison with the Q-construction

Every exact category $\mathscr{C}$ is a Waldhausen category in a natural way, taking the cofibrations to be the admissible monics and $w\mathscr{C} = i\mathscr{C}$ the subcategory of isomorphisms. We hope that our "new" definition of K_i coincides with the old one, defined by Quillen [Qui73, Gra76] via the Q-construction.

We first briefly recall the Q-construction. The reader already familiar with the Q-construction may proceed directly to Subsection 3.3.2, where we show that the $S_\bullet$-construction indeed extends the Q-construction.

3.3.1 An interlude on the Q-construction

The Q-construction takes as input an exact category $\mathscr{C}$ and outputs another category $Q\mathscr{C}$, which is like $\mathscr{C}$ but morphisms are spans.

Definition 3.7. Let $\mathscr{C}$ be an exact category and define $Q\mathscr{C}$ to be the category whose objects are $\mathrm{Ob}\,\mathscr{C}$ and morphisms $Q\mathscr{C}(X, Y)$ are equivalence classes of spans

$$X \twoheadleftarrow Z \hookrightarrow Y$$

where $Z \twoheadrightarrow X$ is an admissible epi and $Z \hookrightarrow Y$ is an admissible mono. Recall that this means there are exact sequences

$$0 \to Z \hookrightarrow Y \to Y' \to 0 \quad \text{and} \quad 0 \to X' \to Z \twoheadrightarrow X \to 0$$

in $\mathscr{C}$, for some Y' and X' in $\mathscr{C}$. Another span $X \twoheadleftarrow Z' \hookrightarrow Y$ is called *equivalent* if there is an isomorphism $Z \to Z'$ in $\mathscr{C}$ making the diamond

$$\begin{tikzcd}
 & Z \arrow[dl, two heads] \arrow[dr, hook] \arrow[dd, "\cong"'] & \\
X & & Y \\
 & Z' \arrow[ul, two heads] \arrow[ur, hook] &
\end{tikzcd}$$

commute. Composition of spans in $Q\mathscr{C}$ is defined using pullbacks: if $X \twoheadleftarrow Z \hookrightarrow Y$ and $Y \twoheadleftarrow V \hookrightarrow W$ are composable spans, then their composition is $X \twoheadleftarrow Z \times_Y V \hookrightarrow W$. This pullback is formed as the diagram

$$\begin{tikzcd}
X & & \\
Z \arrow[u, two heads] \arrow[r, hook] & Y & \\
Z \times_Y V \arrow[u, two heads] \arrow[r, hook] & V \arrow[u, two heads] \arrow[r, hook] & W
\end{tikzcd}$$

in the ambient Abelian category $\mathscr{A}$ containing $\mathscr{C}$. To show that the span $X \twoheadleftarrow Z \times_Y V \hookrightarrow W$ is a well-defined arrow of $Q\mathscr{C}$, we use the fact that $\mathscr{C}$ is closed under extensions in $\mathscr{A}$. Specifically, because $\ker(Z \times_Y V \twoheadrightarrow Z) \cong \ker(V \twoheadrightarrow Y)$, we know the exact sequence

$$0 \to \ker(Z \times_Y V \twoheadrightarrow Z) \to Z \times_Y V \twoheadrightarrow Z \to 0$$

is in $\mathscr{A}$ and the two outer objects are in $\mathscr{C}$, we also know $Z \times_Y V$ is an object of $\mathscr{C}$ (and hence an object of $Q\mathscr{C}$) since $\mathscr{C}$ is closed under extensions. Composition is associative because taking pullbacks is.

Every morphism $X \twoheadleftarrow Z \hookrightarrow Y$ in $Q\mathscr{C}$ can be factored (unique up to isomorphism) into the admissible monic $Z \hookrightarrow Y$ and the (oppositely oriented) admissible epi $X \twoheadrightarrow Z$ by taking one of the arrows in the span to be an equality.

In fact, the assignment that sends $Z \twoheadrightarrow X$ to $X \twoheadleftarrow Z = Z$ and $Z \hookrightarrow Y$ to $Z = Z \hookrightarrow Y$ is functorial and also satisfies the property that if

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d, two heads] & Y \arrow[d, two heads] \\
X \arrow[r, hook] & Z'
\end{tikzcd}$$

is a biCartesian square (both a pullback and a pushout) where all the morphisms are admissible, then the composition

$$(Z = Z \hookrightarrow Y) \circ (X \twoheadleftarrow Z = Z) = X \twoheadleftarrow Z \hookrightarrow Y$$

is equal to the composition

$$(Z' \twoheadleftarrow Y = Y) \circ (X = X \hookrightarrow Z') = X \twoheadleftarrow X \times_{Z'} Y \hookrightarrow Y.$$

Remark 3.8. These two properties actually characterize $Q\mathscr{C}$ in the following sense: We can think of the assignment of $Z \hookrightarrow Y$ to $Z = Z \hookrightarrow Y$ as a covariant functor on the (wide) subcategory of admissible monos in $\mathscr{C}$ (lets call this $\mathscr{C}^{mono}$), landing in $Q\mathscr{C}$. Similarly, the assignment of $Z \twoheadrightarrow X$ to $X \twoheadleftarrow Z = Z$ is a contravariant functor on the (wide) subcategory of admissible epis (call this $\mathscr{C}^{epi}$). Now, suppose we have another category $\mathscr{D}$ and two functors $F_1 \colon \mathscr{C}^{mono} \to \mathscr{D}$, $F_2 \colon (\mathscr{C}^{epi})^{\mathrm{op}} \to \mathscr{D}$ which agree on objects. If for every biCartesian square

$$\begin{tikzcd}
Z \arrow[r, hook, "i"] \arrow[d, two heads, "q"] & Y \arrow[d, two heads, "q'"] \\
X \arrow[r, hook, "i'"] & Z'
\end{tikzcd}$$

we have $F_1(i) \circ F_2(q) = F_2(q') \circ F_1(i')$, then there is a well-defined functor $F \colon Q\mathscr{C} \to \mathscr{D}$ given by $X \mapsto F(X)$ and $X \twoheadleftarrow Z \hookrightarrow Y \mapsto F_1(Z \hookrightarrow Y) \circ F_2(X \twoheadleftarrow Z)$.

The Q-construction came about because Quillen wanted to prove that the "fundamental theorems of K-theory" which were known for K_0 and K_1 also worked for his definition of higher K_n's. The $+$-construction he'd first considered didn't work so well for this, so he came up with the Q-construction, proved everything worked there, and then proved that the two definitions agreed (this is the (in)famous $+ = Q$ theorem).

Since $Q\mathscr{C}$ has a zero object (the same as the zero object of $\mathscr{C}$), its classifying space will be connected with basepoint being the vertex corresponding to 0. In fact, $BQ\mathscr{C}$ is a connected CW complex, and the morphisms $0 = 0 \hookleftarrow X$ and $X \twoheadleftarrow 0 = 0$ give paths from 0 to X in $Q\mathscr{C}$. We can use these paths to connect $BQ\mathscr{C}$ to $K_0(\mathscr{C})$ as defined in Subsection 2.1; we note that the following lemma is a special case of Proposition 3.2 (once we compare the Q- and $S_\bullet$-constructions) but the proof is still helpful to see.

Proposition 3.3. *There is an isomorphism* $\pi_1(BQ\mathscr{C}) \cong K_0(\mathscr{C})$*, and the element of* $\pi_1(BQ\mathscr{C})$ *corresponding to* $[X] \in K_0(\mathscr{C})$ *is represented by the loop* $0 \twoheadleftarrow X \hookrightarrow 0$.

Proof. Recall that $\pi_1(BQ\mathscr{C})$ is the free group on morphisms of $Q\mathscr{C}$ modulo the relations that $[f] * [g] = [f \circ g]$ for every pair of composable morphisms in $Q\mathscr{C}$. We will show that this coincides for the presentation for $K_0(\mathscr{C})$ given in Subsection 2.1.

First note that the composition relation implies that $[0 = 0 \hookrightarrow X] = 1$ in $\pi_1(BQ\mathscr{C})$ for every object X. Moreover, $[Z = Z \hookrightarrow Y] = 1$ in $\pi_1(BQ\mathscr{C})$ because it is part of the composition $[0 = 0 \hookrightarrow Z] \circ [Z = Z \hookrightarrow Y] = [0 = 0 \hookrightarrow Y]$. Since every morphism $X \twoheadleftarrow Z \hookrightarrow Y$ in $Q\mathscr{C}$ can be factored as

$$(X \twoheadleftarrow Z = Z) \circ (Z = Z \hookrightarrow Y),$$

this implies that $[X \twoheadleftarrow Z \hookrightarrow Y] = [X \twoheadleftarrow Z = Z]$ in $\pi_1(BQ\mathscr{C})$. But $X \twoheadleftarrow Z = Z$ similarly appears in the composition

$$(0 \twoheadleftarrow X = X) \circ (X \twoheadleftarrow Z = Z) = (0 \twoheadleftarrow Z = Z).$$

Hence $\pi_1(BQ\mathscr{C})$ is generated by $0 \twoheadleftarrow Z = Z$ which is in 1-to-1 correspondence with $\operatorname{Ob}\mathscr{C}$, the generators of $K_0(\mathscr{C})$. This discussion also shows that $[Z] \in K_0(\mathscr{C})$ is represented by

$$0 \twoheadleftarrow Z \hookrightarrow 0 = (0 \twoheadleftarrow Z = Z) \circ (Z = Z \hookrightarrow 0)$$

in $\pi_1(BQ\mathscr{C})$.

So we just need to show these generators satisfy the correct relations. Suppose $0 \to X \hookrightarrow Y \twoheadrightarrow Z \to 0$ is exact. We can compose $0 = 0 \hookrightarrow Z$ and $Z \twoheadleftarrow Y = Y$ to get $0 \twoheadleftarrow X \hookrightarrow Y$ (because $0 \times_Z Y \cong \ker(Y \twoheadrightarrow Z) \cong X$). Thus the composition relation in $\pi_1(BQ\mathscr{C})$ tells us that

$$[0 \twoheadleftarrow X \hookrightarrow Y] = [Z \twoheadleftarrow Y = Y].$$

We also know $[0 \twoheadleftarrow X \hookrightarrow Y] = [0 \twoheadleftarrow X = X]$ from an earlier observation. Now, factor $0 \twoheadleftarrow Y = Y$ as $(0 \twoheadleftarrow Z = Z) \circ (Z \twoheadleftarrow Y = Y)$ and apply the relation again to see

$$[0 \twoheadleftarrow Y = Y] = [0 \twoheadleftarrow Z = Z] * [Z \twoheadleftarrow Y = Y] = [0 \twoheadleftarrow Z = Z] * [0 \twoheadleftarrow X = X]$$

in $\pi_1(BQ\mathscr{C})$. This is the only relation needed in $\pi_1(BQ\mathscr{C})$ because every $[f] * [g] = [f \circ g]$ can be factored in this way. □

Note that this proof also gives us a way to map morphisms in $Q\mathscr{C}$ to $K_0(\mathscr{C})$, by sending $X \twoheadleftarrow Z \hookrightarrow Y$ to $[\ker(Z \twoheadrightarrow X)]$. This proposition also shows that we can't just define $K_n(\mathscr{C})$ to be the n^{th} homotopy group of $BQ\mathscr{C}$, but we need to shift it by 1, i.e. take the loop space.

Definition 3.9. Let $\mathscr{C}$ be a (small) exact category and define the *K-theory space* of $\mathscr{C}$ to be $K(\mathscr{C}) = \Omega BQ\mathscr{A}$. The *$K$-groups* of $\mathscr{C}$ are the homotopy groups of this space,

$$K_n(\mathscr{C}) = \pi_n K(\mathscr{C}) = \pi_{n+1}(BQ\mathscr{C})$$

for $n \geq 0$. This defines functors $K\colon \mathbf{exCat} \to \mathbf{Top}$ and $K_n\colon \mathbf{exCat} \to \mathbf{Ab}$.

The biproduct $\oplus\colon \mathscr{C}\times\mathscr{C}\to\mathscr{C}$ gives $BQ\mathscr{C}$ a homotopy-commutative H-space structure. In fact, $BQ\mathscr{C}$ is an infinite loop space, and the explicit deloopings are actually given by iterating the Q-construction. That is, for each $n\geq 0$, there are n-fold categories $Q^n\mathscr{C}$ with $\Omega BQ^{n+1}\mathscr{C}\simeq BQ^n\mathscr{C}$. This sequence of $BQ^n\mathscr{C}$ (with $\Omega BQ\mathscr{C}$ at level $n=0$) forms an Ω-spectrum, making $K(\mathscr{C})$ an infinite loop space. More specifics (for $n=2$) are given in [Wei13, Definition IV.6.5 and Exercise IV.6.8]

3.3.2 Comparing the Q-construction and the $S_\bullet$-construction

In this section we will show that there is a homotopy equivalence

$$|iS_\bullet\mathscr{C}|\to BQ\mathscr{C}$$

for an exact category $\mathscr{C}$. This homotopy equivalence passes through $|esd(s_*\mathscr{C})|$, where $s_*\mathscr{C}$ is a certain simplicial set built from $S_*\mathscr{C}$ and esd denotes Segal's edgewise subdivision. We'll also construct a simplicial model $iQ_*\mathscr{C}$ whose realization is equivalent to $BQ\mathscr{C}$, in order to build the following commutative diagram:

$$\begin{tikzcd}
{|s_*\mathscr{C}|} \arrow[r, "\cong"] \arrow[d, "\sim"'] & {|esd(s_*\mathscr{C})|} \arrow[r] \arrow[d, "\sim"] & BQ\mathscr{C} \arrow[d, "\sim"] \\
{|iS_\bullet\mathscr{C}|} \arrow[r, "\cong"'] & {|esd(iS_\bullet\mathscr{C})|} \arrow[r, "\sim"'] & {|iQ_*\mathscr{C}|}
\end{tikzcd}$$

Here, the realization of a simplicial category (e.g. $iQ_*\mathscr{C}$) is the realization of the simplicial space $[n]\mapsto BiQ_n\mathscr{C}$ (and there's secretly some bisimplicial set stuff hanging around). Waldhausen proved that the outermost vertical arrows are equivalences (via the Swallowing Lemma [Wal85, Lemma 1.6.5]) and hence so it the middle one. Our goal is to understand the horizontal maps $esd(s_*\mathscr{C})\to Q\mathscr{C}$ and $esd(S_\bullet\mathscr{C})\to iQ_*\mathscr{C}$ and show the bottom arrow is an equivalence, so then we can conclude the top one is too. This section is the content of [Wal85, Appendix 1.9] and [Wei13, Exercises IV.8.5 and IV.8.6].

Definition 3.10. Let s_* be the simplicial set with $s_n\mathscr{C}=\operatorname{Ob}S_n\mathscr{C}$, so an n-simplex is a n-filtration $0\rightarrowtail A_1\rightarrowtail\dots\rightarrowtail A_n$. The face a degeneracy maps are the object-level maps from the face and degeneracy functors of $S_\bullet\mathscr{C}$.

This means that $esd(s_n\mathscr{C})$ are $(2n+1)$-filtrations $0\rightarrowtail A_1\rightarrowtail\dots\rightarrowtail A_{2n+1}$. The degeneracy map s_j now inserts identities in two places (j and $2n-j$) and the face map d_i omits both the i^{th} and $(2n-i)^{th}$ arrow (unless $i=0,n$; we leave these cases as exercises to the reader).

Now we're going to produce a simplicial map $s_*\mathscr{C}\to Q\mathscr{C}$. To make this easier to understand, we'll use the notation $0\rightarrowtail A'_1\rightarrowtail\dots A'_n\rightarrowtail A_n\rightarrowtail\dots\rightarrowtail A_1\rightarrowtail A_n$ to denote an element of $esd(s_n\mathscr{C})\cong S_{2n+1}\mathscr{C}$. Remember that this filtration also comes with a choice of quotients A_{ij} (we'll use primes to indicate whether the quotient comes from A_i or A'_i). By remembering some of these quotients, and forgetting others, we can produce an element of $N_nQ\mathscr{C}$. For example, an element $0\rightarrowtail A'_1\rightarrowtail A'_2\rightarrowtail A_2\rightarrowtail A_1\rightarrowtail A_0$ in $esd(s_2\mathscr{C})$ gets sent to the element

$$\begin{tikzcd}
A_{0,2'} \arrow[r, hook] \arrow[d, two heads] & A_{0,1'} \arrow[r, hook] \arrow[d, two heads] & A_{0,0'} = A_0 \\
A_{1,2'} \arrow[r, hook] \arrow[d, two heads] & A_{1,1'} & \\
A_{2,2'} & &
\end{tikzcd}$$

of $N_2 Q\mathscr{C}$. The two composable morphisms are the spans $A_{2,2'} \twoheadleftarrow A_{1,2'} \hookrightarrow A_{1,1'}$ and $A_{1,1'} \twoheadleftarrow A_{0,1'} \hookrightarrow A_0$. (Exercise: show this determines a simplicial map!)

Definition 3.11. Let $iQ_n\mathscr{C}$ denote the category with objects $N_nQ\mathscr{C}$ and morphisms are isomorphisms. Then $iQ_*\mathscr{C}$ is a simplicial category with face and degeneracy functors just coming from the usual face and degeneracy maps of the nerve.

The nerve of $Q\mathscr{C}$ is the simplicial set of objects of $iQ_*\mathscr{C}$, just like $s_*\mathscr{C}$ is the simplicial set of objects in $S_\bullet\mathscr{C}$. Since all we've done to get from $s_*\mathscr{C}$ and $N_*Q\mathscr{C}$ to $iS_\bullet\mathscr{C}$ and $iQ_*\mathscr{C}$ is add isomorphisms, the map we defined above determines a morphism of simplicial categories $esd(iS_\bullet\mathscr{C}) \to iQ_*\mathscr{C}$. The last step is the following proposition, which can be shown directly.

Proposition 3.4. *The map $esd(iS_\bullet\mathscr{C}) \to iQ_*\mathscr{C}$ is a levelwise equivalence of categories, $esd(iS_n\mathscr{D}) \xrightarrow{\sim} iQ_n\mathscr{C}$ for each n.*

4 Additivity and other theorems

The additivity theorem says that if $F' \to F \to F''$ is a sequence of exact functors, then K-theory "splits" these functors in the sense that $K(F) = K(F') + K(F'')$. We say a sequence of functors $F' \to F \to F''$ between categories $\mathscr{C} \to \mathscr{D}$ is a *short exact sequence* if

- for exact categories: $0 \to F'(X) \to F(X) \to F''(X) \to 0$ is a short exact sequence in $\mathscr{D}$ for every object $X \in \mathscr{C}$;
- for Waldhausen categories: $F'(X) \rightarrowtail F(X) \twoheadrightarrow F''(X)$ is a cofibration sequence and for every cofibration $X \rightarrowtail Y$ in $\mathscr{C}$, the map $F(X) \cup_{F'(X)} F'(Y) \rightarrowtail F(Y)$ is a cofibration in $\mathscr{D}$.

In the Waldhausen setting, a short exact sequence of functors is sometimes called *cofibration sequence.* Note that when exact categories are viewed as Waldhausen categories, these two notions coincide.

An important example is the extension category $\mathcal{E}(\mathscr{C})$ of exact sequences $E\colon X \rightarrowtail Y \twoheadrightarrow Y/X$ in $\mathscr{C}$. Recall that if $\mathscr{C}$ is exact or Waldhausen, so is $\mathcal{E}(\mathscr{C})$. The functors which are source $s(E) = X$, target $t(E) = Y$, and quotient $q(E) = Y/X$ provide a short exact sequence of functors $s \rightarrowtail t \twoheadrightarrow q$. The extension category is universal in the sense that the data of a an exact sequence of functors $\mathscr{C} \to \mathscr{D}$ is the same as giving an exact functor $\mathscr{C} \to \mathcal{E}(\mathscr{D})$.

Theorem 4.1. *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors between $\mathscr{C} \to \mathscr{D}$ (which are exact or Waldhausen categories). Then $K(F) \simeq K(F') + K(F'')$ as maps of spectra $K(\mathscr{C}) \to K(\mathscr{D})$.*

Hence $K_i(F) = K_i(F') + K_i(F'')$ on homotopy groups. The idea of the proof is to use universality of the extension category to assume $\mathscr{C} = \mathcal{E}(\mathscr{C})$ and then prove the theorem for $s \rightarrowtail t \twoheadrightarrow q$. This is done using the extension theorem ([Wal85, Theorem V.1.3]), which says that $(s, q)\colon \mathcal{E}(\mathscr{C}) \to \mathscr{C} \times \mathscr{C}$ induces a homotopy equivalence on the K-spaces, with homotopy inverse given by the coproduct $(X, Y) \mapsto (X \rightarrowtail X \coprod Y \twoheadrightarrow Y)$. We note that it suffices to show this just assuming $\mathscr{C}$ and $\mathscr{D}$ are Waldhausen, but there is another proof for exact categories which uses Quillen's Theorem A (see [Wei13, p.367–368]).

Waldhausen uses the Additivity Theorem to obtain an explicit delooping of $K(\mathscr{C})$ as the iterated $S_\bullet$-construction. Also from additivity he deduces other important theorems in higher algebraic K-theory, such as the localization and approximation theorem. We summarize these theorems in Subsection 4.2.

4.1 The Additivity Theorem

There are several equivalent statements of Waldhausen's additivity theorem ([Wal85, Proposition 1.3.4]), many of which involve the extension category which we defined in Subsection 2.2.3.

If $\mathscr{A}$ and $\mathscr{C}$ are Waldhausen subcategories of $\mathscr{B}$, then we can define the *extension category of $\mathscr{C}$ by $\mathscr{A}$* to be the Waldhausen subcategory of $\mathcal{E}(\mathscr{B})$ consisting of cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ with $A \in \mathscr{A}$ and $C \in \mathscr{C}$. Note that $\mathcal{E}(\mathscr{C}) = \mathcal{E}(\mathscr{C}, \mathscr{C}, \mathscr{C})$.

Recall that an *cofibration sequence of functors* $F' \rightarrowtail F \rightarrowtail F''$ between Waldhausen categories $\mathscr{C} \to \mathscr{D}$ means there are natural transformations $F' \Rightarrow F \Rightarrow F''$ of exact functors so that (i) for every $C \in \mathscr{C}$,

$$F'(C) \rightarrowtail F(C) \twoheadrightarrow F''(C)$$

is a cofibration sequence in $\mathscr{D}$, and (ii) for every cofibration $A \rightarrowtail B$ in $\mathscr{C}$, the square of cofibrations

$$\begin{tikzcd}
F'(A) \arrow[r, tail] \arrow[d, tail] & F'(B) \arrow[d, tail] \\
F(A) \arrow[r, tail] & F(B)
\end{tikzcd}$$

has the property that $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$ is a cofibration.

Example 4.2. Let $s, t, q\colon \mathcal{E}(\mathscr{C}) \to \mathscr{C}$ be the functors which take a cofibration sequence $A \rightarrowtail B \twoheadrightarrow C$ to A, B, and C, respectively. Then $s \rightarrowtail t \twoheadrightarrow q$ is a cofibration sequence of functors.

The category $\mathcal{E}(\mathscr{C})$ is universal in the sense that specifying a cofibration sequence of functors $F' \rightarrowtail F \twoheadrightarrow F''\colon \mathscr{C} \to \mathscr{D}$ is the same as specifying an exact functor $G\colon \mathscr{C} \to \mathcal{E}(\mathscr{D})$. (Take $F' = sG$, $F = tG$, $F'' = qG$, etc.)

Theorem 4.3. *The following are equivalent:*

(i) The projection

$$wS_\bullet\mathcal{E}(\mathscr{A},\mathscr{C},\mathscr{B}) \to wS_\bullet\mathscr{A} \times wS_\bullet\mathscr{B}$$

which sends $A \rightarrowtail C \twoheadrightarrow B$ *to* (A, B) *is a homotopy equivalence.*

(ii) The projection

$$wS_\bullet\mathcal{E}(\mathscr{C}) \to wS_\bullet\mathscr{C} \times wS_\bullet\mathscr{C}$$

which sends $A \rightarrowtail C \twoheadrightarrow B$ *to* (A, B) *is a homotopy equivalence.*

(iii) The two maps

$$wS_\bullet\mathcal{E}(\mathscr{C}) \to wS_\bullet\mathscr{C}$$

mapping $A \rightarrowtail C \twoheadrightarrow B$ *to* C, $A \vee B$ *are homotopic.*

(iv) If $F' \rightarrowtail F \twoheadrightarrow F''$ *is a cofibration sequence of functors* $\mathscr{C} \to \mathscr{D}$, *then there is a homotopy*

$$|wS_\bullet F| \simeq |wS_\bullet F'| \vee |wS_\bullet F''| (= |wS_\bullet(F' \vee F'')|).$$

Waldhausen ultimately proves that (ii) holds ([Wal85, Theorem 1.4.2]), thus showing the other formulations of additivity hold as well. We discuss his proof in the next subsection.

Remark 4.4. We can induct on the number of exact functors in our sequence to get something like an Euler characteristic. A sequence $0 \to X_n \to \cdots \to X_0 \to 0$ in a category $\mathscr{C}$ is *admissibly exact* if each map decomposes as $X_{n+1} \to Y_n \to X_n$ and each $Y_n \rightarrowtail X_n \twoheadrightarrow Y_{n-1}$ is exact. A sequence of functors $0 \to F_0 \to \dots F_n \to 0$ is admissibly exact if it lands in admissibly exact sequences. For such a sequence of functors, the Additivity Theorem (and induction) implies that $\sum_j (-1)^j K(F^j) = 0$ as maps $K_i(\mathscr{C}) \to K_i(\mathscr{D})$ for all $i \geq 0$.

Waldhausen also uses additivity to get a delooping theorem in [Wal85, §1.5], showing that $|wS_\bullet\mathscr{C}| \simeq \Omega|S_\bullet S_\bullet\mathscr{C}|$. In more detail, given an exact functor $F\colon \mathscr{C} \to \mathscr{D}$, construct $S_\bullet(F)$ as the pullback over $S_\bullet\mathscr{C} \to S_\bullet\mathscr{D} \leftarrow PS_\bullet\mathscr{D}$ (where $PS_n\mathscr{D} = S_{n+1}\mathscr{D}$). This defines a simplicial Waldhausen category so that. all the functors in the defining diagram are exact. The key claim is that

$$wS_\bullet\mathscr{D} \to wS_\bullet S_\bullet(F) \to wS_\bullet S_\bullet\mathscr{C}$$

is a homotopy fiber sequence. The maps in this sequence are induced by maps $\mathscr{D} \to S_\bullet(F) \to S_\bullet\mathscr{C}$, where $S_\bullet F$ to $S_\bullet\mathscr{C}$ is the evident projection and $\mathscr{D} \to S_\bullet(F)$ comes from considering $\mathscr{D}$ as a trivial simplicial category. To prove this claim, Waldhausen applies additivity to show that

$$wS_\bullet\mathscr{D} \to wS_\bullet S_n(F) \to wS_\bullet S_n\mathscr{C}$$

is a fibration up to homotopy for each n. Specifically, apply version (iv) of the additivity theorem to a cofibration sequence

$$j' \rightarrowtail \mathrm{id} \twoheadrightarrow j''$$

of endofunctors on $S_n(F)$, where $j'(S_n(F)) = \mathscr{D}$ and $j''(S_n(F)) = S_n\mathscr{C}$. Additivity implies $wS_\bullet\mathscr{D} \times wS_\bullet S_n\mathscr{C} \to wS_\bullet S_n(F)$ is a homotopy equivalence,and hence the sequence above is split exact up to homotopy.

Now, in the case when F is the identity $\mathscr{C} \to \mathscr{C}$, then $wS_\bullet S_\bullet F = P(wS_\bullet S_\bullet\mathscr{C})$ is contractible (because $P(wS_\bullet S_\bullet\mathscr{C})$ is augmented over $|wS_0\mathscr{C}| = *$), and hence $\Omega|wS_\bullet S_\bullet\mathscr{C}| \xrightarrow{\sim} |wS_\bullet\mathscr{C}|$. In general, $|wS_\bullet^{(n)}\mathscr{C}| \xrightarrow{\sim} \Omega|wS_\bullet^{(n+1)}\mathscr{C}|$ for every $n \geq 1$, hence we get a Ω-spectrum.

4.1.1 Waldhausen's proof

We summarize Waldhausen's proof that (ii) holds. Define $s_n\mathscr{C} = \operatorname{Ob} S_n\mathscr{C}$ to be the set of length-n filtrations. So if we take $\mathscr{C}^w(m)$ to be the category of length-m compositions of weak equivalences, then $s_nC^w(m)$ is (isomorphic to) $wN_mS_n\mathscr{C}$. In particular, $|[m] \mapsto s_*\mathscr{C}^w(m)| \cong |wS_\bullet\mathscr{C}|$. Thus it suffices to prove

$$s_*\mathcal{E}\mathscr{C} \xrightarrow{\sim} s_*\mathscr{C} \times s_*\mathscr{C},$$

because then we would have levelwise equivalences $s_*\mathcal{E}(\mathscr{C}^w(m)) \xrightarrow{\sim} s_*\mathscr{C}(m) \times s_*\mathscr{C}(m)$ which would then geometrically realize to an equivalence (mumble-mumble Reedy cofibrant). So most of the proof of additivity is proving the claim in the display above, which notably only makes use of the category with cofibrations structure and not the weak equivalences.

There are a couple of useful lemmas about $s_*\mathscr{C}$. First, [Wal85, Lemma 1.4.1] says that this s_* construction is functorial and an isomorphism of functors induces a homotopy after s_*. As a consequence, an exact equivalence of categories (with cofibrations) induces an equivalence after s_* and in particular this means $s_*\mathscr{C}$ is homotopy equivalent to $iS_\bullet\mathscr{C}$, where $\mathscr{C}$ is a category with cofibrations and $w\mathscr{C}$ is taken to be isomorphisms $iso(\mathscr{C})$.

In order to prove $s_*\mathcal{E}(\mathscr{C}) \to s_*\mathscr{C} \times s_*\mathscr{C}$ is an equivalence, Waldhausen uses simplicial versions of Quillen's Theorems A and B.

Lemma 4.5. *Let $f\colon X_* \to Y_*$ be a map of simplicial sets and $y \in Y_n \cong sSet(\Delta^n, Y)$. Define a simplicial set $f/(n,y)_*$ as the pullback*

$$\begin{tikzcd}
f/(n,y)_* \arrow[r] \arrow[d] & X_* \arrow[d, "f"] \\
\Delta^n_* \arrow[r, "y"'] & Y_*
\end{tikzcd} .$$

A. If $f/(n,y)$ is contractible for every (n,y) then f is a homotopy equivalence.

B. If for every $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$ and every $y \in Y_n$, the induced map $f/(m,\phi^*y) \to f/(n,y)$ is an equivalence, then for every (n,y) the pullback diagram above is a homotopy pullback.

In the context of additivity, f will be the map $s_*s\colon s_*\mathcal{E}(\mathscr{C}) \to s_*\mathscr{C}$ which sends a cofibration sequence $A \rightarrowtail B \twoheadrightarrow C$ to the source A. The idea is to show that s_*s satisfies the hypothesis of B, and consider the resulting diagram at $n = 0$,

$$\begin{tikzcd}
f/(0,y)_* \arrow[r] \arrow[d] & X_* \arrow[d, "f"] \\
* \arrow[r, "y"'] & Y_*
\end{tikzcd} .$$

In this case, $f/(0,y)_*$ looks like $s_*\mathcal{E}'(\mathscr{C})$, where $\mathcal{E}'(\mathscr{C}) \subseteq \mathcal{E}(\mathscr{C})$ consists of cofibration sequences $* \rightarrowtail B \twoheadrightarrow C$. Note that $B \twoheadrightarrow C$ must be an isomorphism, so in fact $f/(0,y)_* \cong s_*\mathscr{C}$.

This means

$$\begin{tikzcd}
s_*\mathscr{C} \arrow[r] & s_*\mathcal{E}(\mathscr{C}) \arrow[r] & s_*\mathscr{C} \\
B \arrow[r, mapsto] & * \rightarrowtail B = B & \\
& A \rightarrowtail B \twoheadrightarrow C \arrow[r, mapsto] & A
\end{tikzcd}$$

is a homotopy fiber sequence. There is a map of fiber sequences

$$\begin{tikzcd}
s_*\mathscr{C} \arrow[r] \arrow[d, equal] & s_*\mathscr{C} \times s_*\mathscr{C} \arrow[r] \arrow[d] & s_*\mathscr{C} \arrow[d, equal] \\
s_*\mathscr{C} \arrow[r] & s_*\mathcal{E}(\mathscr{C}) \arrow[r] & s_*\mathscr{C}
\end{tikzcd}$$

where the middle arrow sends $(A, B) \mapsto (A \rightarrowtail A \coprod B \twoheadrightarrow B)$. Since the maps on the fiber and base are identities, this middle map must be an equivalence. But the middle map is a section to $s \vee q$, the map we wanted to show is an equivalence. Voilà!

So the only thing left to show is that we may apply B to $s_*s\colon s_*\mathcal{E}(\mathscr{C}) \to s_*\mathscr{C}$. That is, we need to show that for every $\phi\colon [m] \to [n]$ and $y \in s_n\mathscr{C}$, the map $\phi_*\colon f/(m, \phi^* y) \to f/(n, y)$ is an equivalence. In fact, it suffices to show it for maps $[0] \to [n]$, since a map $[m] \to [n]$ will sit inside a triangle $[0] \to [m] \to [n] = [0] \to [n]$. So we just need to show

$$\phi^i_*\colon f/(0, *) \to f/(n, y)$$

is an equivalence, where $y \in s_n\mathscr{C}$ and $\phi^i\colon [0] \to [n]$ sends $*$ to i, and a proof can be found on [Wal85, p.338–340].

4.1.2 McCarthy's proof

We now summarize a different proof that (ii) holds, due to McCarthy [McC93]. The first step, just as in Waldhausen's proof, is to reduce to just considering the simplicial set $s_*\mathscr{C}$ rather than $S_\bullet\mathscr{C}$. The main method used in the proof is also a version of Quillen's Theorem A, but one specifically formulated for the $S_\bullet$-construction.

Given an exact functor $F\colon \mathscr{C} \to \mathscr{D}$, define a bisimplicial set $s_{*,*}F|\mathscr{D}$ where $s_{n,m}F|\mathscr{D}$ is the pullback of $s_m\mathscr{C} \xrightarrow{s_nF} s_m\mathscr{D} \leftarrow s_{n+m}\mathscr{D}$, where the function $s_{n+m}\mathscr{D} \to s_m\mathscr{D}$ sends an object

$$* \rightarrowtail d_1 \rightarrowtail \ldots \rightarrowtail d_m \rightarrowtail e_0 \rightarrowtail e_1 \rightarrowtail \ldots \rightarrowtail e_n$$

(suppressing choices of quotients in the notation) to $* \rightarrowtail d_1 \rightarrowtail \ldots \rightarrowtail d_m$ (again suppressing choices of quotients). The bisimplicial structure maps are obtained from those of $s_*\mathscr{D}$ and the definition as a pullback. For ease of notation, we will denote an object as above by $d_* \rightarrowtail e_*$. Explicitly, the objects of the pullback are pairs $(c_*, d_* \rightarrowtail e_*)$ with $Fc_* = d_*$.

Remark 4.6. The simplicial category $S_\bullet F$ discussed earlier fixes $n = 1$; explicitly, $\operatorname{Ob} S_mF = s_{1,m}F|\mathscr{D}$.

Note that there are two projection maps $s_{n,m}F|\mathscr{D} \to s_m\mathscr{C}$ and $s_{n,m}F|\mathscr{D} \to s_n\mathscr{D}$ which send $(c_*, d_* \rightarrowtail e_*)$ to c_* and e_*/e_0, respectively. (Here e_*/e_0 means $* \rightarrowtail e_1/e_0 \rightarrowtail \ldots \rightarrowtail e_n/e_0$.) These assemble into maps of bisimplicial sets $\pi\colon s_{*,*}F|\mathscr{D} \to s_*\mathscr{C}$ and $\rho\colon s_{*,*}F|\mathscr{D} \to s_*\mathscr{D}$, where the targets are considered as bisimplicial sets which are trivial in one direction. The map of simplicial sets $\rho_n\colon s_{*,n}F|\mathscr{D} \to s_n\mathscr{D}$ (where the set $s_n\mathscr{D}$ is viewed as a constant simplicial set) admits a section i_n which sends $* \rightarrowtail e_1 \rightarrowtail \ldots \rightarrowtail e_n$ to $(c_*, d_* \rightarrowtail e_*)$ where both c_* and d_* consist of a sequence (of the appropriate length) of identities on the zero object. Let $E_n\colon s_{*,n}F|\mathscr{D} \to s_{*,n}F|\mathscr{D}$ denote the composite $i_n \circ \rho_n$.

Theorem 4.7. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be an exact functor between Waldhausen categories. Then the following are equivalent:*

- *(i) the map of simplicial sets $s_*F\colon s_*\mathscr{C} \to s_*\mathscr{D}$ induces a homotopy equivalence after geometric realization,*
- *(ii) the map of bisimplicial sets $\rho\colon s_{*,*}F|\mathscr{D} \to s_*\mathscr{D}$ induces a homotopy equivalence after geometric realization.*

Moreover, (ii) holds if the map of simplicial sets $E_n\colon s_{*,n}F|\mathscr{D} \to s_{*,n}F|\mathscr{D}$ is homotopic to the identity.

Proof sketch. The proof that (i) and (ii) are equivalent is essentially the same strategy as the proof of Quillen Theorem A. In particular, there is a diagram of bisimplicial sets

$$\begin{tikzcd}
s_*\mathscr{C} \arrow[d, "s_*F"'] & s_{*,*}F|\mathscr{D} \arrow[l, "\pi"'] \arrow[r, "\rho"] \arrow[d] & s_*\mathscr{D} \arrow[d, equal] \\
s_*\mathscr{D} & s_{*,*}\operatorname{id}_{\mathscr{D}}|\mathscr{D} \arrow[l, "\pi"] \arrow[r, "\rho"'] & s_*\mathscr{D}
\end{tikzcd}$$

and one can show that both arrows labeled π and the bottom arrow labeled ρ are always equivalences after geometric realization. Hence, by 2-of-3, s_*F is a homotopy equivalence if and only if the top arrow labeled ρ is.

For the second claim, note that in order to show that ρ is an equivalence, it suffices to show that each ρ_n is. Since i_n is a section of ρ_n, it suffices to show that the composite $E_n := i_n \circ \rho_n$ is homotopic to the identity. $\square$

For the purposes of additivity (specifically to prove formulation (ii)), we want to apply this theorem to the exact functor $(s, q)\colon \mathcal{E}(\mathscr{C}) \to \mathscr{C} \times \mathscr{C}$ which sends $A \rightarrowtail B \twoheadrightarrow C$ to the pair (A, C). We will check that in this case the maps E_n are all homotopic to the identity.

Explicitly, in this case, E_n sends an element of $s_{*,n}(s,q)|\mathscr{C}^2$ such as

$$\left[\begin{array}{c}
\begin{tikzcd}
* \arrow[r, tail] \arrow[d, tail] & A_1 \arrow[r, tail] \arrow[d, tail] & \dots \arrow[r, tail] & A_m \arrow[d, tail] \\
* \arrow[r, tail] \arrow[d, two heads] & B_1 \arrow[r, tail] \arrow[d, two heads] & \dots \arrow[r, tail] & B_m \arrow[d, two heads] \\
* \arrow[r, tail] & C_1 \arrow[r, tail] & \dots \arrow[r, tail] & C_m
\end{tikzcd} \\
\hline
\begin{tikzcd}
* \arrow[r, tail] & A_1 \arrow[r, tail] & \dots \arrow[r, tail] & A_m \arrow[r, tail] & S_0 \arrow[r, tail] & \dots \arrow[r, tail] & S_n \\
* \arrow[r, tail] & C_1 \arrow[r, tail] & \dots \arrow[r, tail] & C_m \arrow[r, tail] & T_0 \arrow[r, tail] & \dots \arrow[r, tail] & T_n
\end{tikzcd}
\end{array}\right]$$

(omitting from the notation the choices of quotients) to the element

$$\left[\begin{array}{c}
\begin{tikzcd}
* \arrow[r, tail] \arrow[d, tail] & * \arrow[r, tail] \arrow[d, tail] & \dots \arrow[r, tail] & * \arrow[d, tail] \\
* \arrow[r, tail] \arrow[d, two heads] & * \arrow[r, tail] \arrow[d, two heads] & \dots \arrow[r, tail] & * \arrow[d, two heads] \\
* \arrow[r, tail] & * \arrow[r, tail] & \dots \arrow[r, tail] & *
\end{tikzcd} \\
\hline
\begin{tikzcd}
* \arrow[r, tail] & * \arrow[r, tail] & \dots \arrow[r, tail] & * \arrow[r, tail] & S_0/S_0 \arrow[r, tail] & \dots \arrow[r, tail] & S_n/S_0 \\
* \arrow[r, tail] & * \arrow[r, tail] & \dots \arrow[r, tail] & * \arrow[r, tail] & T_0/T_0 \arrow[r, tail] & \dots \arrow[r, tail] & T_n/T_0
\end{tikzcd}
\end{array}\right].$$

To show this map is homotopic to the identity, McCarthy's idea is to instead consider the subcategory $\mathcal{E}'(\mathscr{C}) \subseteq \mathcal{E}(\mathscr{C})$ of cofibration sequences whose source is $*$ and the commutative diagram

$$\begin{tikzcd}
s_*\mathcal{E}'(\mathscr{C}) \arrow[r, hook] \arrow[dr, "{(s',q')}"'] & s_*\mathcal{E}(\mathscr{C}) \arrow[d, "{(s,q)}"] \\
 & s_*\mathscr{C} \times s_*\mathscr{C}
\end{tikzcd}$$

where the unlabeled top horizontal arrow is induced by the inclusion. There is a retraction $s_*\mathcal{E}(\mathscr{C}) \to s_*\mathcal{E}'(\mathscr{C})$ induced by the functor which sends $A \rightarrowtail B \twoheadrightarrow C$ to $* \rightarrowtail C = C$. (Technically, the functor only a retraction up to natural isomorphism, but after geometric realization this natural isomorphism induces a homotopy to the identity.) This commutative diagram induces a commutative diagram of simplicial sets

$$\begin{tikzcd}
s_{*,n}(s', q')|\mathscr{C}^2 \arrow[r, hook] \arrow[d, "E'_n"'] & s_{*,n}(s, q)|\mathscr{C}^2 \arrow[l, bend right, "\Gamma"'] \arrow[d, "E_n"] \\
s_{*,n}(s', q')|\mathscr{C}^2 \arrow[r, hook] & s_{*,n}(s, q)|\mathscr{C}^2 \arrow[l, bend right, "\Gamma"']
\end{tikzcd}$$

for all $n \geq 0$, where the unlabeled hooked arrows are induced by the inclusion $\mathcal{E}'(\mathscr{C}) \subseteq \mathcal{E}(\mathscr{C})$ and Γ is the map induced by the retraction. In McCarthy's notation, $s_{*,n}(s', q')|\mathscr{C}^2 = X$. Hence, to show that E_n is homotopic to the identity, it suffices to show that (1) Γ is homotopic to the identity and (2) E'_n is homotopic to the identity.

Claim (2) is easier than claim (1). For (2), note that the map E'_n is just like the map E_n described previously but with $A_i = *$ for all $1 \leq i \leq m$. Hence E'_n "does nothing" in the first row that appears below the horizontal line so we only need to think about what's happening in the second row. Here, we see that E'_n is given by

$$(* \rightarrowtail C_1 \rightarrowtail \ldots \rightarrowtail C_m \rightarrowtail T_0 \rightarrowtail \ldots \rightarrowtail T_n) \mapsto (* \rightarrowtail * \rightarrowtail \ldots \rightarrowtail * \rightarrowtail T_0/T_0 \rightarrowtail \ldots \rightarrowtail T_n/T_0)$$

and there's a simplicial homotopy from this map to the identity by taking successive quotients (first by C_1, then by C_2, and so on up to T_0). The simplicial homotopy for (1) makes explicit use of the fact that Waldhausen categories admit pushouts along cofibrations, as McCarthy details on [McC93, p.328].

Remark 4.8. Examining Waldhausen's and McCarthy's proofs closely, we see that the proofs are actually not all that different. The main difference is that Waldhausen uses simplicial versions of Quillen's Theorems while McCarthy uses versions specifically for the $S_\bullet$-construction.

4.2 Some other theorems for K-theory

Many of the fundamental theorems of K-theory let us compute the K-theory of something by replacing it with something else, hopefully something simpler. These theorems were already established for the lower K-groups, and so any good higher K-theory should also obey these rules. In addition to the Additivity Theorem, there are (at least) five other important theorems: Cofinality, Approximation, Resolution, Devissage, and Localization. We will state but not prove these theorems. Everything we say is covered in more detail in [Wei13, Chapter V].

4.2.1 Cofinality

Recall that a category $\mathscr{D}$ is *cofinal* in another category $\mathscr{C}$ if for all objects $X \in \mathscr{C}$ there is another object $X' \in \mathscr{C}$ so that $X \coprod X'$ (or $\otimes$ or $\oplus$ or whatever symbol is appropriate) is in $\mathscr{D}$. A functor $F\colon \mathscr{D} \to \mathscr{C}$ is called cofinal if its image is cofinal in $\mathscr{C}$. So for instance f.g. free modules are cofinal in f.g. projective modules because every projective module can be direct summed with another projective module to get a free one.

The Cofinality Theorem says that the K-theory of $\mathscr{D}$ is the same as the K-theory of $\mathscr{C}$ in certain conditions. These conditions are slightly different for symmetric monoidal categories, exact categories, and Waldhausen categories.

Theorem 4.9 (Cofinality for symmetric monoidal categories)**.** *Suppose $F\colon \mathscr{C} \to \mathscr{D}$ is a cofinal symmetric monoidal functor. If $Aut_{\mathscr{C}}(X) \cong Aut_{\mathscr{D}}(FX)$ for all $X \in \mathscr{C}$, then the basepoint components of $K(\mathscr{C})$ and $K(\mathscr{D})$ are equivalent, hence $K_n(\mathscr{C}) \cong K_n(\mathscr{D})$ for all $n \geq 1$.*

The proof shows the $\mathbb{Z}$-homology of the basepoint components (which can be computed as the colimit of the homology of $BAut(X) \cong BAut(FX)$) is the same, and hence they must be homotopy equivalent. Note that this theorem only guarantees equality on K_n for $n \geq 1$; in general $K_0(\mathscr{C})$ may not be the same as $K_0(\mathscr{D})$.

Example 4.10. We know that $\mathbf{F}(R)$ is cofinal in $\mathbf{P}(R)$, but $K_0(\mathbf{P}(R)) = K_0(R)$ is not equal to $K_0(\mathbf{F}(R)) \cong \mathbb{Z}$ in general. Since F is just the inclusion of the full subcategory $\mathbf{F}(R)$ into $\mathbf{P}(R)$, the automorphisms are the same in both categories and so the Cofinality Theorem applies. Thus $K_n(\mathbf{P}(R)) \cong K_n(\mathbf{F}(R))$ for all $n \geq 1$.

Theorem 4.11 (Cofinality for Waldhausen categories)**.** *Suppose $\mathscr{D}$ is a cofinal Waldhausen subcategory of $\mathscr{C}$, closed under extensions. If $K_0(\mathscr{D}) = K_0(\mathscr{C})$ then $wS_\bullet\mathscr{D} \to wS_\bullet\mathscr{C}$ and $K(\mathscr{D}) \to K(\mathscr{C})$ are homotopy equivalences. In particular, $K_n(\mathscr{D}) \cong K_n(\mathscr{C})$ for all $n \geq 0$.*

This also gives us a cofinality theorem for exact categories, considering them as Waldhausen categories.

4.2.2 Localization and Approximation

These are two theorems in Waldhausen K-theory which are technical but important for computations. We already stated the K_0 version of localization way back in Subsection 2.1. The generalized version is a bit harder to state and would require me to introduce certain technical axioms (namely, the cylinder functor and cylinder axiom). The reader unfamiliar with the conditions in the theorem statement can replace them with “some technical axioms” (or, more helpfully, read the definitions in [Wei13, Chapter IV]).

Let $\mathscr{C}$ be a category with cofibrations and two subcategories of weak equivalences, $v\mathscr{C} \subseteq w\mathscr{C}$, both of which give $\mathscr{C}$ a Waldhausen structure. Let $\mathscr{C}^w$ denote the Waldhausen subcategory of $(\mathscr{C}, v\mathscr{C})$ consisting of all $X \in \mathscr{C}$ for which $0 \rightarrowtail X$ is in $w\mathscr{C}$.

Theorem 4.12. *Suppose $(\mathscr{C}, w\mathscr{C})$ is a Waldhausen category. If $w\mathscr{C}$ satisfies the Saturation and Extension axioms and $(\mathscr{C}, w)$ satisfies the Cylinder Axiom, then*

$$K(\mathscr{C}^w) \to K(\mathscr{C}, v\mathscr{C}) \to K(\mathscr{C}, w\mathscr{C})$$

is a homotopy fibration. In particular, there is a long exact sequence of K-groups.

The proof uses the bicategory $v_*w_*\mathscr{C}$ whose horizontal maps are in $w\mathscr{C}$ and vertical maps are in $v\mathscr{C}$. The proof is a bit technical.

The historically most important of localization is to relate the K-theory of exact categories and their bounded chain complexes. This is the Gillet-Waldhausen theorem.

Theorem 4.13. *Let $\mathscr{C}$ be exact, closed under kernels of surjections in an Abelian category. Then the exact inclusion $\mathscr{C} \to \mathbf{Ch}^b(\mathscr{C})$ induces a homotopy equivalence $K(\mathscr{C}) \to K(\mathbf{Ch}^b(\mathscr{C}))$. In particular, $K_n(\mathscr{C}) \cong K_n(\mathbf{Ch}^b(\mathscr{C}))$ for all $n \geq 0$.*

The Approximation Theorem, also for Waldhausen categories, lets us compare K-theory across categories when we can "approximately lift" morphisms.

Definition 4.14. Let $F\colon \mathscr{C} \to \mathscr{D}$ be an exact functor. We say F satisfies the *approximate lifting property* if for any map $F(X) \to Y$ in $\mathscr{D}$, there is a map $X \to X'$ in $\mathscr{C}$ and a weak equivalence $F(X') \xrightarrow{\sim} Y$ in $\mathscr{D}$, i.e.

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, "F"'] & X' \arrow[d, "F"] & \\
F(X) \arrow[r] \arrow[rr, bend right] & F(X') \arrow[r, dashed, "\sim"] & Y
\end{tikzcd}.$$

Theorem 4.15. *Suppose $F\colon \mathscr{C} \to \mathscr{D}$ is an exact functor between saturated Waldhausen categories. Suppose F satisfies the approximate lifting property and moreover*

(i) $f \in w\mathscr{C}$ iff $F(f) \in w\mathscr{D}$,

(ii) $\mathscr{C}$ satisfies the Cylinder Axiom.

Then $wS_\bullet\mathscr{C} \to wS_\bullet\mathscr{D}$ and $K(\mathscr{C}) \to K(\mathscr{D})$ are homotopy equivalences. In particular, $K_n(\mathscr{C}) \cong K_n(\mathscr{D})$ for all $n \geq 0$.

A corollary of the Approximation Theorem is that changing cofibrations may not affect K-theory: if $(\mathscr{C}, co\mathscr{C}, w\mathscr{C})$ is a Waldhausen category (satisfying saturation and the cylinder axiom) with another Waldhausen structure coming from a subcategory of cofibrations $co\mathscr{C} \subseteq \tilde{co}\mathscr{C}$, then the inclusion of Waldhausen categories $(\mathscr{C}, co\mathscr{C}, w\mathscr{C}) \subseteq (\mathscr{C}, \tilde{co}\mathscr{C}, w\mathscr{C})$ induces an equivalence on K-theory (see [Wei13, V.2.5.1]).

4.2.3 Resolution and Devissage

The resolution and devissage theorems are similar in the sense that they let us compute the K-theory of an exact/Abelian category in terms of an exact/Abelian subcategory which objects of the larger category admit finite filtrations of objects in the smaller one. We'll state the Resolution Theorem first. Let $\mathscr{A}$ be a full exact subcategory of an exact category $\mathscr{B}$ so that $\mathscr{A}$ is closed under extensions and kernels of admissible epis in $\mathscr{B}$.

Theorem 4.16. *Suppose every object X of $\mathscr{B}$ has a finite $\mathscr{A}$-resolution*

$$0 \to A_n \to \cdots \to A_1 \to A_0 \to X \to 0.$$

Then $K(\mathscr{A}) \simeq K(\mathscr{B})$ and thus $K_n(\mathscr{A}) \cong K_i(\mathscr{B})$ for all $n \geq 0$.

According to Weibel (cf. [Wei13, Remark V.3.1.2]) it is not known (at least at the time of his writing) how to extend this for Waldhausen categories in general. But there is one for Walhausen subcategories of chain complexes (see the Thomason-Trobaugh Resolution Theorem [Wei13, Theorem V.3.9]).

The Devissage (which comes from the French word for "unscrewing") is very similar to the resolution theorem. Let $i\colon \mathscr{A} \subseteq \mathscr{B}$ be an inclusion of Abelian categories so that $\mathscr{A}$ is an exact subcategory of $\mathscr{B}$ which is closed under subobjects (monics into) and quotients.

Theorem 4.17. *If every object X of $\mathscr{B}$ has a finite filtration*

$$0 = X_k \hookrightarrow \cdots \hookrightarrow X_1 \hookrightarrow X_0 = X$$

of objects in $\mathscr{B}$ such that every subquotient X_i/X_{i-1} lives in $\mathscr{A}$. Then $K(\mathscr{A}) \simeq K(\mathscr{B})$ and so $K_n(\mathscr{A}) \cong K_n(\mathscr{B})$.

The proof uses Quillen Theorem A (see [Wei13, Section V.4]). The applications of this theorem are mostly to compute K-theory for various types of rings with lots of adjectives in front of them (e.g. Artinian, local, semisimple, Noetherian). Since the publication of the K-book, there is also now a version of Devissage for Waldhausen categories, see [Rap22].

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