UPENN GRAD PIZZA SEMINAR

SOCIAL CHOICE PROBLEMS & ALGEBRAIC TOPOLOGY

OR: WHY YOU CAN'T ALWAYS GET WHAT YOU WANT (ACCORDING TO MATH)

MAXINE E. CALLE

SOCIAL CHOICE PROBLEMS

Suppose you have N people and a space of preferences P

WHO SHOULD BE THE U.S. PRESIDENT?

P = {CANDIDATES}

DISCRETE SET

SOCIAL CHOICE PROBLEMS

Suppose you have N people and a *space of preferences* P

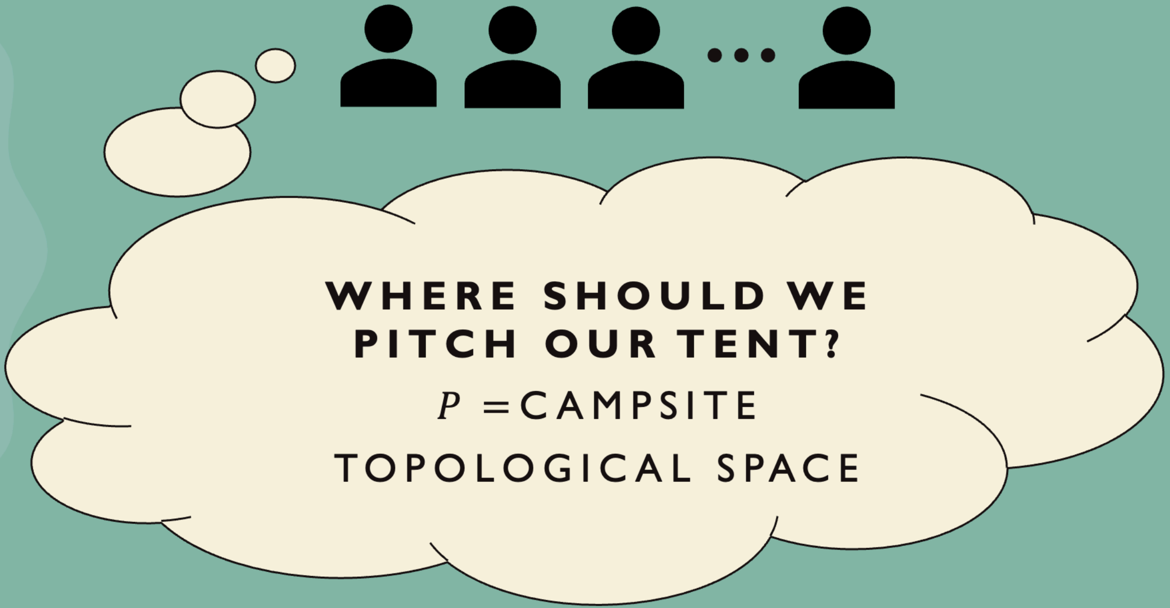
WHAT'S THE BEST REAL NUMBER?

$P = \mathbb{R}$

METRIC SPACE

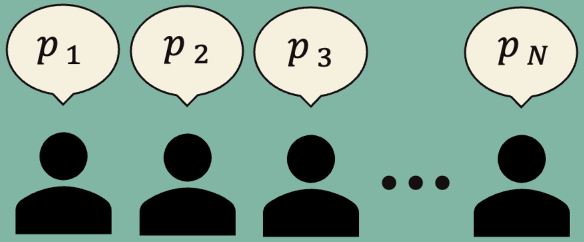
SOCIAL CHOICE PROBLEMS

Suppose you have N people and a *space of preferences* P



SOCIAL CHOICE PROBLEMS

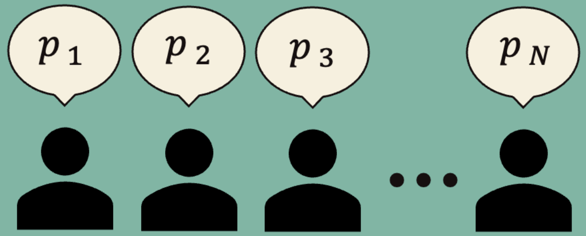
Suppose you have N people and a *space of preferences* P



Each person picks a *preference* $p_k \in P$

Given $(p_1, p_2, \ldots, p_N) \in P^N$, how do you choose a "fair" $p \in P$?

HOW DO WE COMPROMISE?



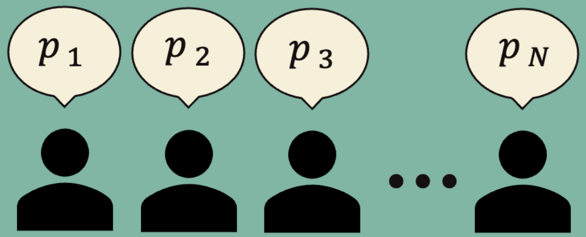
Given $(p_1, p_2, \ldots, p_N) \in P^N$, how do you choose a "fair" $p \in P$?

DEFINITION A *social choice function* $F: P^N \to P$ satisfies:

1. Continuity: F is a continuous map

SMALL CHANGES IN PREFERENCES DON'T CHANGE THE OUTCOME VERY MUCH

HOW DO WE COMPROMISE?



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DEFINITION A *social choice function* $F : P^N \to P$ satisfies:

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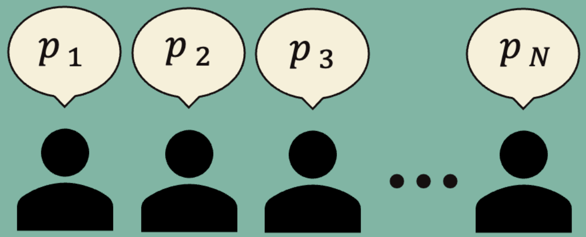
2. Unanimity: $F(p, p, \ldots, p) = p$

NO SURPRISES

HOW DO WE COMPROMISE?



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3. Anonymity: $F(p_1, p_2, \ldots, p_N) = F(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(N)})$ for all $\sigma \in \Sigma_N$

EVERYONE'S VOTE IS EQUAL

HOW DO WE COMPROMISE?

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WHO SHOULD BE THE PRESIDENT?

P = {CANDIDATES}

- VOTING SYSTEMS

WHAT'S THE BEST NUMBER?

$P = \mathbb{R}$

- ARITHMETIC MEAN
- GEOMETRIC MEAN

WHERE SHOULD OUR TENT GO?

P = CAMPSITE

- ???

IS THERE EVEN A POSSIBLE F?

WHEN CAN WE COMPROMISE?

<u>DEFINITION</u> A *social choice function* $F: P^N \to P$ satisfies:

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WHERE SHOULD OUR TENT GO?

P = CAMPSITE

- ???

IS THERE EVEN A POSSIBLE F?

Algebraic topology can help (surprise!)

ALGEBRAIC TOPOLOGY???

TOPOLOGY

TOPOLOGICAL SPACES + CONTINUOUS MAPS

⇒

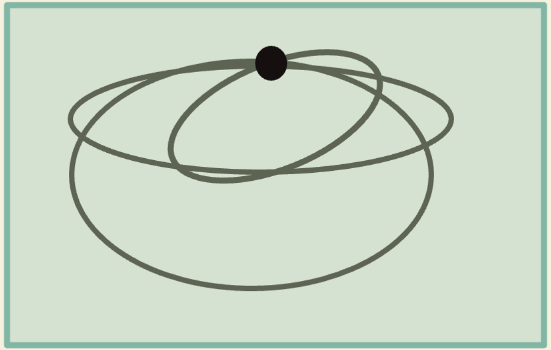
ALGEBRA

GROUPS + HOMOMORPHISMS

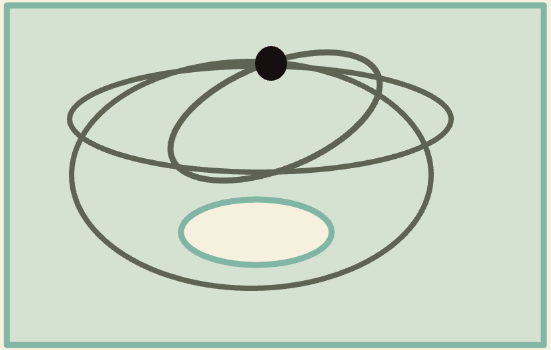
CATEGORY THEORY

CATEGORIES + FUNCTORS

EXAMPLE The fundamental group $\pi_1(X) = \{\text{loops in } X\}/\sim$



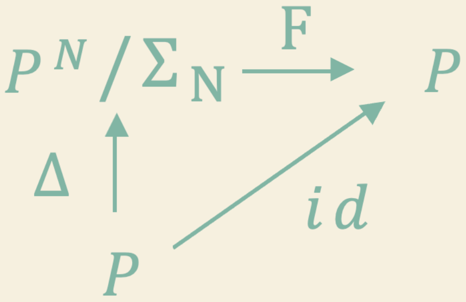
$\pi_1 \mapsto e$



$\pi_1 \mapsto \mathbb{Z}$

ALGEBRAIC TOPOLOGY

TOPOLOGY → **ALGEBRA**



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ALGEBRAIC TOPOLOGY

TOPOLOGY $\xrightarrow{\ \pi_1\ }$ **ALGEBRA**

$$\begin{array}{ccc} P^N/\Sigma_N & \xrightarrow{F} & P \\ \Delta \uparrow & \nearrow_{id} & \\ P & & \end{array}$$

$$\begin{array}{ccc} \pi_1(P)^N/\Sigma_N & \xrightarrow{\pi_1(F)} & \pi_1(P) \\ \Delta \uparrow & \nearrow_{id} & \\ \pi_1(P) & & \end{array}$$

"N-mean"

Does P have a social choice function for $N \geq 2$ people? $\xrightarrow{\ \pi_1\ }$ Does $\pi_1(P)$ have an N-mean?

ALGEBRAIC TOPOLOGY

Does P have a social choice function for $N \geq 2$ people?

π_1 →

Does $\pi_1(P)$ have an N-mean?

<u>Theorem</u> if G has an N-mean for $N \geq 2$, then G is abelian and every element of G is divisible by N.

←

$\rightsquigarrow$ There's no N-mean on $\mathbb{Z}$ for any $N \geq 2$.

←

Suppose a CW complex P admits a social choice function for every N.

<u>Theorem</u> the only finitely generated G which has an N-mean for every $N \geq 2$ is $G = e$.

...But there are non-finitely generated examples

<u>Theorem</u> If P is a finite complex, then $P \simeq *$

Otherwise, $P = \prod K(\mathbb{Q}, k_i)$ is a product of Eilenberg-MacLane spaces

↑ these spaces make no one happy: $\forall d > 0\ \exists (p_1, \ldots, p_N)$ with $|F(p_1, \ldots, p_N) - p_i| > d$

THANKS FOR LISTENING!

PLEASE COME ASK ME QUESTIONS ABOUT ANYTHING
OR YOU CAN EMAIL ME: CALLEM@SAS.UPENN.EDU