"ALGEBRAIC K-THEORY OF ORBISPACES" EXPLANATION FOR A GENERAL AUDIENCE

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Introduction

Theoretical mathematics research is like solving a challenging puzzle: it begins with what is already known, identifies a question no one has answered, and then carefully finds a solution. Unlike some other fields, a theoretical math thesis usually does not rely on experiments or surveys. Instead, it uses logical reasoning and proofs, i.e. step-by-step explanations that show why results are true. Although the symbols and formulas may look daunting, the research process is creative and exploratory.

This document gives an introduction and overview of my Ph.D. thesis that is intended to be accessible to any reader. A central requirement of any math Ph.D. thesis is that it must contain new, original results. My thesis tells the story — the background, the discoveries, the reasoning, and what might come next — of a particular research problem that I worked on during my Ph.D. program, in an area of mathematics called *algebraic topology*. The main focus of my work is to take a well-known construction, called *algebraic K-theory*, and figure out how to generalize it to apply to a new kind of mathematical object called an *orbispace*.

Topological invariants

Topology is the mathematical study of shapes — both familiar shapes like circles and cubes, and also complicated, higher-dimensional shapes that are difficult to visualize. Unlike in geometry, topologists do not keep track of rigid measurements like distance, angle, or size. Two shapes are considered to be "topologically the same" if one can be obtained from the other by squishing, stretching, or other elastic deformations.

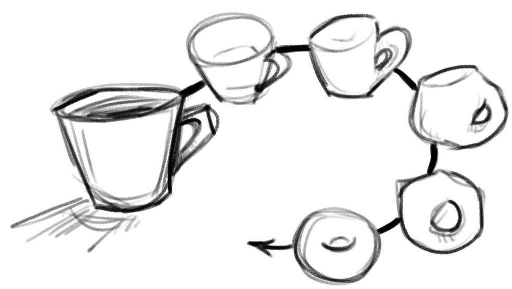

FIGURE 2.1. A coffee mug is "topologically the same" as a donut

By centering flexible, topological features rather than rigid, geometric ones, we are led to equate shapes that we would normally think of as distinct. There is a classic joke among mathematicians that a topologist cannot tell the difference between a coffee mug and a donut, because a (suitably malleable) coffee mug could be molded into a donut without creating any rips or tears. As another example, there are 26 unique letters in the English alphabet, but only nine topologically unique ones (depending on the choice of font):

A R	D O	K H	E F T Y	B	P	Q	X	C G I J L M N S U V W Z

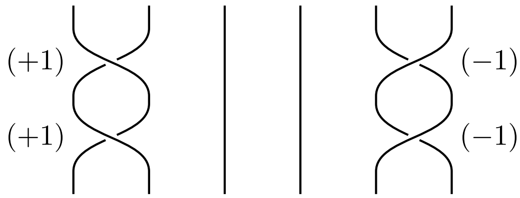

Figure 2.2. A topological invariant of DNA is whether it is (+)-supercoiled, relaxed, or (−)-supercoiled

In general, it can be very difficult to tell whether two given shapes are topologically the same or not. This problem has prompted the development of many different kinds of mathematics, each with its own set of techniques and tools. Within the context of *algebraic topology*, we use measurements called *invariants* to distinguish topological shapes. Just as a function sends inputs to outputs, a topological invariant takes in a shape as input and outputs something algebraic, like a number, a collection of numbers, a formula, or a more abstract mathematical structure. If two shapes are topologically the same, then they will produce the same output. Conversely, if two shapes produce different outputs, they must be topologically distinct.

Topological invariants have found applications in a variety of disciplines, including in *topological data analysis*, where data scientists use invariants to extract meaning from large data sets, and in the study of the *topology of DNA*, where biochemists and molecular biologists study topological three-dimensional structures of the double helix.

Counting holes

Although topological invariants behave like functions, they are often far more difficult to describe than something like $f(x) = 2x + 3$. Nonetheless, many invariants are founded upon a simple idea, like counting how many times a DNA strand crosses over or under itself. One particularly important idea in algebraic topology is to count how many "holes" a shape has, since topological deformations do not change this number. A donut's hole is fundamental to its topological nature, and any other shape that is topologically equivalent to it, like a coffee mug (with a handle), must share this feature.

However, not all holes are created equal. An inflated balloon also has a hole, its interior, but we would not expect to be able to deform it into a donut in a topological way. An algebraic topologist would argue that the two shapes are fundamentally different because the donut's hole is one-dimensional and the balloon's is two-dimensional.

In general, we can make sense of n-dimensional holes for every every non-negative whole number $n = 0, 1, 2, 3, \ldots$. For each n, there is an invariant called the n^{th} *Betti number* which essentially counts the number of n-dimensional holes in a shape — the 0^{th} Betti number just counts how many "pieces" the shape has. The 0^{th}, 1^{st}, and 2^{nd} Betti numbers of the donut are 1, 1, and 0, while for a balloon they are 1, 0, and 1. All the higher Betti numbers are 0, since the donut and balloon both naturally live in three-dimensional space. Because the two shapes have different Betti numbers, they cannot be topologically the same.

Another important invariant is called the *Euler characteristic*, which is a single whole number that can be obtained from the Betti numbers by taking an alternating sum: the 0^{th} Betti number minus the 1^{st}, plus the 2^{nd}, minus the 3^{rd}, and so on. The Euler characteristic of the donut is $1 - 1 + 0 = 0$, while the Euler characteristic for the balloon is $1 - 0 + 1 = 2$. Hence the Euler characteristic also detects the difference between these shapes.

If two shapes are topologically the same, then they must have the same Euler characteristic, but the converse is not necessarily true. Two shapes may share the same Euler characteristic while being topologically distinct. For example, imagine taking a donut, slicing it into two halves, and then gluing those pieces back together to get two smaller donuts as in Figure 3.1.

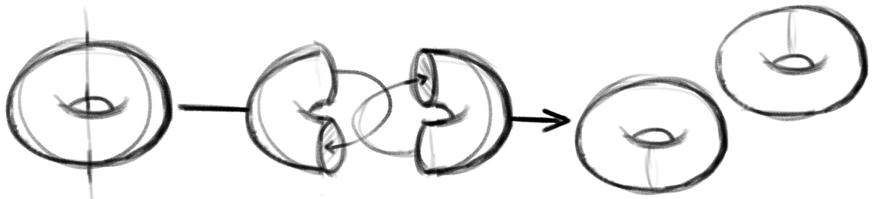

FIGURE 3.1. A donut becomes two donuts

This is certainly not a topological process, as we cut up the donut into pieces. Indeed, the two shapes cannot be topologically the same, because they have different numbers of holes: the first three Betti numbers of a donut are 1, 1, and 0, while for two donuts they are 2, 2, and 0. However, the Euler characteristic has not changed, since $1 - 1 + 0 = 2 - 2 + 0 = 0$.

This example demonstrates that the Euler characteristic is sensitive to how shapes decompose into smaller pieces. That is, two shapes have the same Euler characteristic if they can be constructed from the same topological building blocks.

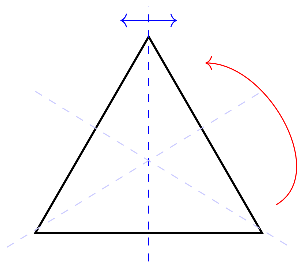

FIGURE 3.2. An equilateral triangle has both reflectional (blue) and rotational (red) symmetries

SYMMETRIES

Topological invariants become even more interesting when the shapes come with some sort of inherent symmetry. A *symmetry* of a shape is a transformation — such as a reflection, rotation, or translation — so that the end result looks exactly the same as the start.

Every shape has a "trivial" symmetry where nothing moves, but many familiar shapes come with other, more interesting transformations. For example, an equilateral triangle has five non-trivial symmetries (combinations of rotations and reflections), an isosceles triangle has one (a reflection), while a scalene triangle has none at all. Although all these triangles are topologically the same, they are differentiated by their inherent symmetries.

To adapt the Betti numbers and the Euler characteristic to this setting, we no longer just want to count holes, but rather *symmetric holes*. In an isosceles triangle, there are two different kinds of one-dimensional symmetric holes: one that is centered on the axis of reflection (in blue) and another (in red) that consists of two holes which are swapped by the reflection symmetry.

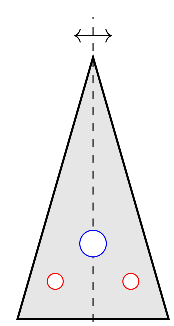

FIGURE 4.1. An isosceles triangle with two different symmetric holes

Now, rather than having a Betti number for each dimension $n = 0, 1, 2, 3, \ldots$, there is a Betti number for every different kind of symmetric hole in each dimension. The Euler characteristic similarly needs to incorporate symmetry information. The more complicated the symmetries of the shape are, the more complicated these invariants become.

FIGURE 4.2. Illustration of an orbispace

This dissertation project focuses on certain kinds of shapes, called *orbispaces*, whose symmetries can be very intricate. In the context of orbispaces, we are interested in not only *global symmetries*, like reflections and rotations, but also *local symmetries* which are only visible upon restricting our attention to a certain part of the shape. In particular, a given orbispace may not have any non-trivial global symmetries, but many interesting local ones.

Adapting topological invariants like the Betti numbers and the Euler characteristic to this localized setting is a complicated task. In fact, it is only within the last few decades that the tools from algebraic topology have become fine-grained enough to apply to orbispaces, granting access to invariants like the Euler characteristic. This dissertation aims to take advantage of these developments to generalize a certain invariant, called *algebraic K-theory*, to account for the local symmetries present in an orbispace.

DETECTING HIDDEN COMPLEXITY WITH ALGEBRAIC K-THEORY

The fundamental philosophy of algebraic K-theory is that mathematical objects can be understood by how they break apart into smaller pieces — much like how molecules decompose into atoms. This simple idea has surprisingly powerful applications in many fields of mathematics, and continues to fuel interesting research since its development in the 1960s.

This thesis is particularly concerned with an approach to algebraic K-theory developed by Friedhelm Waldhausen in the late 20th century. Waldhausen's theory studies the ways topological spaces can be built, decomposed, and related to one another. It keeps track not just of the final shape, but also the processes and choices involved in constructing it, and asks when a complicated transformation can be undone or simplified.

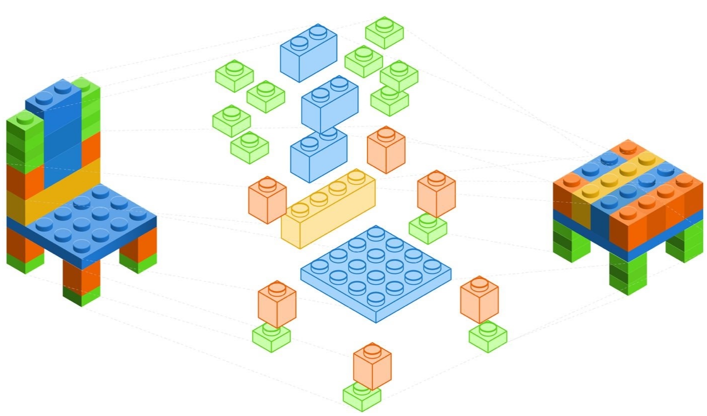

FIGURE 5.1. The same LEGO pieces can be used to build a chair and a table

For instance, imagining that our topological spaces were built from LEGOs, we could use the same set of LEGO blocks to build all kinds of different shapes, as in Figure 5.1. Or we could use LEGOs to build two castles that look identical from the outside, but on the inside one uses clever internal scaffolding and the other does not. Waldhausen's algebraic K-theory is an invariant which keeps track of "organizational complexity": it tells you which LEGOs were used to construct your shapes, how the shapes were assembled, and which assemblies are genuinely different even if the end result looks the same.

These ideas are closely related to the study of *cobordisms* and Stephan Smale's award-winning work from the 1960s. A cobordism can be thought of as a movie which describes how a shape changes over time. If we lined up each frame from the movie next to each other, in chronological order, we would obtain a higher-dimensional object that records the entire process at once. The shape at the beginning of the movie forms one boundary of this object, and the shape at the end forms another.

In two dimensions, the cobordism corresponding to a single object splitting into two has a particularly vivid form, shown in Figure 5.2. Starting with one circle and ending with two circles, the combined movie traces out a surface that looks like a pair of pants: one "waist hole" at the top, two "leg holes" at the bottom, and a smooth transition in between. Each horizontal slice represents a moment in time, showing how one loop gradually pinches and separates into two.

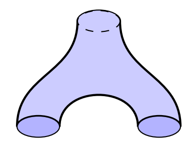

FIGURE 5.2. The "pair of pants" cobordism (which is not an h-cobordism)

An h-cobordism is a particularly well-behaved cobordism in which the intermediate space is, from the standpoint of topology, essentially the same as either boundary. At first glance, this notion suggests that nothing interesting is happening in the middle, but topology is full of examples that seem simple but actually have hidden complicated structure. Smale's h-cobordism theorem gives criteria for when these cobordisms are in fact uninteresting. Waldhausen's work showed that algebraic K-theory is a natural language that explains a deeper structure underlying Smale's result.

Overview of Dissertation

There are many mathematical dialects which can be used to talk about orbispaces. The framework used in this dissertation is called *homotopy theory*, which combines intuition with a high degree of abstraction. Chapter 1 lays out the necessary language to talk about orbispaces and compares this theory to the existing, well-understood theory of spaces with (global) symmetries. Chapter 2 introduces the concept of an *orbispectrum*, which is a particularly structured kind of orbispace — this extra structure is a lot more complicated and requires more bookkeeping to keep track of. One of the results of Chapter 2 is that an orbispectrum can be thought of as a different kind of object called a *Mackey functor*. Mackey functors are still complicated objects, but are somewhat easier to construct and study, particularly from the perspective of higher algebraic K-theory.

The second part of my thesis focuses on generalizing higher algebraic K-theory to this context. In Chapter 3, I describe how to use algebraic K-theory to construct an orbispectrum from relatively simple data. Finally, in Chapter 4, I introduce the most important construction of my thesis: the *algebraic K-theory of an orbispace*. In this chapter, I confirm that this construction encodes interesting (and expected) topological invariants of orbispaces. For example, just as classical algebraic K-theory generalizes the Euler characteristic, so does my construction generalize a known analogue of the Euler characteristic for orbispaces. In future work, I hope to better understand the relationship between my K-theory construction and cobordisms of orbispaces, where the extra structure of the local symmetries is accounted for.