"ALGEBRAIC K-THEORY OF ORBISPACES" EXPLANATION FOR A MATHEMATICAL AUDIENCE

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Introduction

My Ph.D. thesis is motivated by the development and study of *homotopy invariants for orbifolds*, particularly those invariants arising from higher algebraic K-theory. Algebraic K-theory has proved to be a powerful framework connecting a wide area of mathematical disciplines since its development in the late 20th century. While originally defined to capture algebraic invariants of rings, higher algebraic K-theory has since grown far beyond its initial scope to encompass increasingly rich and intricate settings. One powerful example that is particularly relevant to my work is Waldhausen's *algebraic K-theory of spaces* [51], also called A-theory. Waldhausen's motivation was to develop a space-level lift of Smale's h-cobordism theorem [46], which Smale developed as a crucial ingredient of his Fields medal work on the resolution of the higher-dimensional generalized Poincaré conjecture. Waldhausen's vision was realized several decades later by the stable parametrized h-cobordism theorem [55], the proof of which marked the conclusion of a long development in geometric topology. Later work further solidified the importance of Waldhausen's constructions in accessing information about differential topology [1, 15, 28, 31, 57].

Orbifolds are generalizations of manifolds that allow for certain singularities. The term orbifold stands for "orbit-manifold," and was coined by Thurston in the context of 3-manifolds [49], although the idea appeared in earlier work of Satake on automorphic forms [43]. Orbifolds arise naturally as moduli spaces encoding families of objects with symmetries, and manifest in a variety of mathematical areas as well as other fields like physics and even music theory [3, 50]. However, many classical techniques from manifold topology do not immediately apply to orbifolds, and there is still much to be understood about how to extend important tools to this singular setting. For instance, celebrated work of Thom [48] uses stable homotopy theory to show that certain characteristic numbers detect when a smooth manifold is the boundary of another manifold. On the other hand, determining when an orbifold is the boundary of another remains an open problem in general [5, 23, 24]. Similarly, foundational results such as Poincaré Duality have only been developed for orbifolds in certain settings [2, 19].

Orbifolds are inherently equivariant objects, as they carry built-in symmetries arising from group actions. Hence a natural toolkit comes from equivariant algebraic topology, the study of algebraic invariants that respect these symmetries. This area has seen remarkable advances in recent years driven by the resolution of the famous Kervaire Invariant One problem [29, 34] and the disproof of the Telescope Conjecture [14]. The idea that equivariant homotopy theory (and more generally *global* homotopy theory, which accounts for symmetries across all groups at once) should have powerful applications to the homotopy theory of orbifolds appears in work of Gepner–Henriques [27] and Schwede [45], and was recently implemented by Pardon [41] in a Pontryagin–Thom isomorphism for orbifolds. Using this toolkit, my thesis extends Waldhausen's algebraic K-theory of spaces to the orbifold setting. The techniques used draw on my expertise in equivariant homotopy theory, particularly in the context of equivariant algebraic K-theory [18, 16, 17], and rely on modern developments in infinity categories and parametrized homotopy theory [36, 11, 40, 21]. The following sections provide more motivation and background for the thesis work; the reader who is interested in only the main results may proceed directly to Section 5.

Motivation: h-cobordism theorems and algebraic K-theory

A central problem in topology is the classification of manifolds, but determining when two given smooth manifolds M and N are diffeomorphic is an extraordinarily difficult problem. Since every diffeomorphism is a homotopy equivalence, a natural question to ask is when a given homotopy equivalence $\phi\colon M \to N$ actually arises from a diffeomorphism. A classical instance of this problem is the generalized Poincaré conjecture, which asks whether a closed n-manifold homotopy equivalent to the n-sphere must necessarily be homeomorphic or diffeomorphic to it. This conjecture was resolved (for homeomorphisms) in high dimensions, $n \geq 5$, by Smale in the 1960s [47, 46], and Smale's approach was to reframe the classification problem in geometric terms via the *h-cobordism theorem*.

The idea is to turn the homotopy equivalence $\phi\colon M \to N$ into a certain kind of cobordism W_ϕ, called an h-cobordism, so that ϕ is homotopic to a diffeomorphism if and only if W_ϕ is diffeomorphic to the cylinder $M \times I$. Thus, upgrading homotopy equivalences to diffeomorphisms reduced to determining when an h-cobordism is trivial. Smale showed that if M^n is simply connected, for $n \geq 5$, then every h-cobordism on M is trivial, hence every ϕ must be homotopic to a diffeomorphism. For non-simply connected manifolds, the s-cobordism theorem of Barden, Mazur, and Stallings [10, 39] classifies h-cobordisms via the algebraic invariant *Whitehead torsion*. In particular, an h-cobordism is trivial precisely when its Whitehead torsion vanishes.

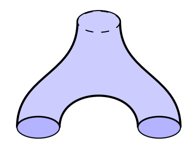

Figure 2.1. The "pair of pants" cobordism is not an h-cobordism

The Whitehead torsion takes values in the *Whitehead group*, which may be described as a quotient of the first algebraic K-group $K_1(\mathbb{Z}[\pi_1(M)])$. The appearance of K_1 in this theorem motivated much of Waldhausen's foundational work on the algebraic K-theory of spaces [53, 54, 52, 51]. Waldhausen's work lifts Whitehead torsion from an algebraic invariant to a homotopy-theoretic one and culminates in the stable parametrized h-cobordism theorem [55], which relates the space of (stable) h-cobordisms on M to a K-theory space $A(M)$, recovering the original s-cobordism theorem on connected components [31, 51]. In this way, the algebraic K-theory of spaces naturally encodes the higher structures of h-cobordism theory, and it is this interaction between manifold topology and higher algebraic K-theory that motivates the work in this thesis.

From manifolds to orbifolds

This thesis is a first step towards extending h-cobordism theorems to *orbifolds*, which generalize smooth manifolds by allowing certain singularities. Just as manifolds are locally modeled on $\mathbb{R}^n$, orbifolds are locally modeled on quotients $\mathbb{R}^n/G$ by actions of finite groups. Orbifolds are ubiquitous in many areas of mathematics, including geometric topology, algebraic geometry, and mathematical physics.

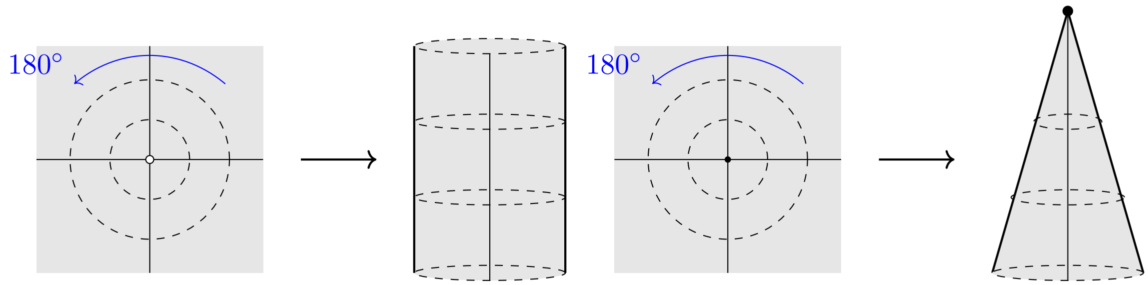

FIGURE 3.1. The group $C_2 = \{\pm 1\}$ acts freely on the punctured plane $\mathbb{C} \setminus \{0\}$ by multiplication, and the orbit space is the infinite cylinder $S^1 \times \mathbb{R}$. On all of $\mathbb{C}$, the C_2-action has one fixed point (the origin) which becomes a *cone point* with isotropy C_2.

One source of examples comes from group actions on manifolds. If a finite group G acts freely and properly discontinuously on a manifold M, then the orbit space M/G inherits the structure of a smooth manifold. When the action is not free, M/G is an orbifold: points with nontrivial stabilizers become "orbifold points;" see Figure 3.1. Such examples called *global quotients*.

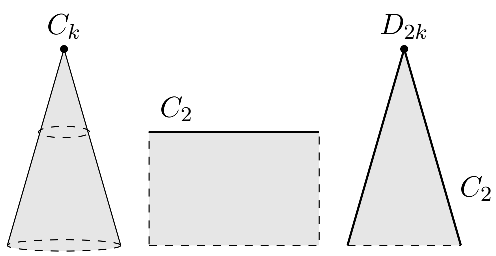

FIGURE 3.2. 2-dimensional orbifolds can have cone points, mirror points, and corner points

More generally, orbifolds can be described via charts M/G, where the manifold M and the finite group G may vary. For example, in dimension two, the local charts are of the form $\mathbb{R}^2/G$ where $G \leq O(2)$ is a finite subgroup, hence G may be (i) a cyclic rotation group C_k, whose orbits give a *cone point* of angle $2\pi/k$; (ii) a reflection group C_2 of order 2, whose orbits give *mirror points*; or (iii) a dihedral group D_{2k} of order $2k$, whose orbits give *corner points* of angle π/k. A similar classification works for the local behavior of three-dimensional orbifolds [22].

MOTIVATING EXAMPLE: ORBIFOLD EULER CHARACTERISTICS

Orbifolds may be distinguished from one another using a version of the Euler characteristic, which may be a rational number. For instance, a global quotient M/G will satisfy $\chi(M/G) = \frac{1}{|G|}\chi(M)$. More generally, the *Euler–Satake characteristic* of an orbifold X is given by

$$\chi(X) = \sum_{\sigma \subseteq X} (-1)^{\dim(\sigma)} \frac{1}{|G_\sigma|},$$

where the sum ranges over (open) cells $\sigma \subseteq X$ whose local isotropy group is G_σ, according to some appropriate cell decomposition of X. Just as every compact, smooth manifold can be triangulated, every compact orbifold admits such an appropriate cell structure, and this formula does not depend on the chosen decomposition of X [44].

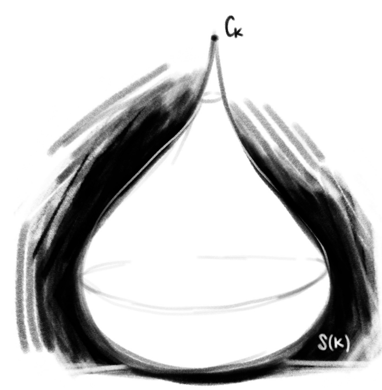

FIGURE 4.1. The Euler–Satake characteristic of the teardrop $S(k)$ is $1 + \frac{1}{k}$

Unlike for classical surfaces, the Euler–Satake characteristic does not classify 2-dimensional compact orbifolds. For example, as shown in Figure 4.2, the torus $T^2 = S^1 \times S^1$ with Σ_2-action $(x, y) \mapsto (y, x)$ yields the global quotient orbifold $T^2//\Sigma_2$, whose

Euler–Satake characteristic is $\chi(T^2//\Sigma_2) = \frac{1}{2}\chi(T^2) = 0$, the same as the usual torus. Indeed, any global quotient of T^2 will have orbifold Euler characteristic 0.

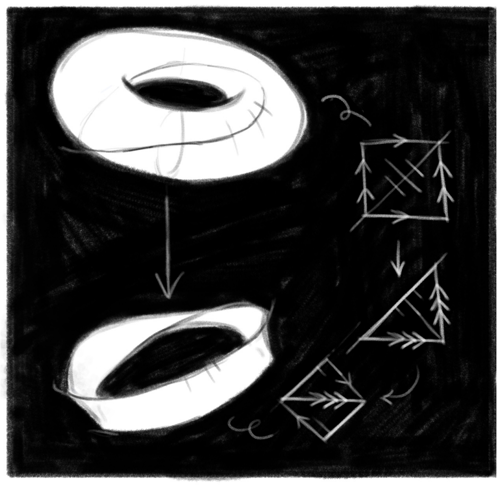

FIGURE 4.2. The global quotient $T^2//\Sigma_2$ is a Mobius strip with mirror points along the boundary.

Moreover, the Euler–Satake characteristic is not uniquely determined by the property of being *additive* (meaning that it takes disjoint unions to sums), unlike in the classical case. Indeed, many distinct additive invariants exist [7, 30, 33, 26], many of which have the same flavor of the Euler–Satake characteristic but are not clearly derived from it in general.

To address this second issue, a universal version of the Euler characteristic for orbifolds was introduced in [26] which records isotropy groups explicitly. However, it is not valued in $\mathbb{Z}$, nor even in $\mathbb{Q}$; instead, it is valued in a free Abelian group $\mathcal{A}_{\mathcal{O}}$ generated by isomorphism classes of finite groups. The *universal Euler characteristic* of an orbifold X is given by

$$\chi_{\mathcal{O}}(X) = \sum_{\sigma \subseteq X} (-1)^{\dim(\sigma)}[G_\sigma] \in \mathcal{A}_{\mathcal{O}},$$

again according to some appropriate cell decomposition of X. For example, the universal Euler characteristic of the teardrop $S(k)$ of Figure 4.1 is $[e]+[C_k]$. The previous orbifold Euler characteristic is obtained from this one by sending $[G] \mapsto \frac{1}{|G|}$. The descriptor "universal" is indicative of the fact that any additive invariant of orbifolds factors through $\chi_{\mathcal{O}}$ in a unique way.

The appearance of the group $\mathcal{A}_{\mathcal{O}}$ is naturally explained by algebraic K-theory. The universality of the usual Euler characteristic is encoded by a group isomorphism $K_0(\mathrm{CW}_*^{\mathrm{fin}}) \cong \mathbb{Z}$, where the left side is the zeroth algebraic K-group of finite pointed CW complexes. In the orbifold setting, we replace finite pointed CW complexes with their orbi-analogues which are constructed from cells indexed both by dimension and isotropy group. There is then an isomorphism $K_0(\mathrm{orbiCW}_*^{\mathrm{fin}}) \cong \mathcal{A}_{\mathcal{O}}$ which exhibits $\chi_{\mathcal{O}}$ as universal among additive homotopy invariants of orbifolds.

The universal Euler characteristic $\chi_{\mathcal{O}}$ highlights that orbifold invariants naturally live in rich algebraic structures that keep track of isotropy groups and their interactions. To systematically study such invariants, we use the homotopy theoretic framework of *orbispaces* (which are like orbifolds without the smooth structure on local charts) and their stable analogues, *orbispectra*.

Overview and summary of main results

Chapter 1 lays the foundation for the study of orbispaces. My treatment follows that of Gepner–Henriques [27], except we use the framework of ∞-categories (as in [20]). Most of the content of this chapter is not original, although the precise formulations of certain results may not be stated as such in the literature. The main original contribution of Chapter 1 is a generally applicable theory of (Bredon) cohomology for orbispaces which, to my knowledge, has previously only been considered for global quotients [2, 42]. Bredon cohomology is like the analogue of cellular cohomology for orbispaces and obeys expected axioms like functoriality, homotopy invariance, and the suspension isomorphism.

My definition closely mirrors the original definition by Bredon [13] in the G-equivariant setting, and when $X = M//G$ is a global quotient, then I recover the G-equivariant Bredon cohomology of M.

Just as ordinary (cellular) cohomology with coeffients in an Abelian group is represented by an Eilenberg–Maclane spectrum, an analogous result holds for Bredon cohomology. In this setting, Abelian groups are replaced with *globally-defined Mackey functors* (sometimes called *biset functors*), which were developed to provide a unifying framework for phenomena in representation theory [12, 56]. A globally-defined Mackey functor $\underline{A}$ is specified by a collection of Abelian groups $\underline{A}(G)$, one for each finite group G, that are compatible with standard operations from subgroup inclusions, such as induction and restriction. Examples include representation rings and group cohomology.

Theorem 5.1. *If $\underline{A}$ is a globally-defined Mackey functor, then cohomology with coefficients in $\underline{A}$ is represented by the Eilenberg–Maclane orbispectrum of $\underline{A}$.*

The notion of orbispectrum mentioned above is taken from the study of global homotopy theory [45, 20]. Key to this result is my work in Chapter 2, which provides a framework for understanding orbispectra as globally-defined Mackey functors valued in spectra (rather than Abelian groups).

Theorem 5.2. *Orbispectra may be modeled as spectral globally-defined Mackey functors. In particular, an orbispectrum is specified by a collection of spectra $\underline{E} = \{E(G)\}_G$, one for each finite group G, along with restriction, transfer, and conjugation structure maps connecting them. For each $n \geq 0$, the collection of homotopy groups $\pi_n\underline{E} = \{\pi_n(E(G))\}_G$ naturally form a globally-defined Mackey functor in Abelian groups.*

This result is useful because spectral Mackey functors may be studied using tools from category theory. For instance, in Chapter 3, I use this framework to show that orbispectra arise from K-theory constructions whose input are categorical versions of globally-defined Mackey functors.

Theorem 5.3. *There is an algebraic K-theory functor that produces orbispectra from suitable categorical globally-defined Mackey functors. All connective orbispectra arise in this way, up to equivalence.*

The utility of this theorem is to provide a recipe for producing the complicated structure of an orbispectrum from simpler categorical data. For instance, the Eilenberg–Maclane orbispectrum of a globally-defined Mackey functor M may be obtained by considering M as a "one-object" categorical Mackey functor and applying K-theory. The sphere orbispectrum $\mathbb{S}_{\mathcal{O}}$ is obtained as the K-theory of an orbi-version of finite sets, giving an analogue of the Barratt–Priddy–Quillen Theorem.

This K-theory construction also lays the foundation for some of the work in Chapter 4, where I consider the analogue of Waldhausen's algebraic K-theory for orbispaces. Given an orbispace X, one could always forget the orbispace structure and simply consider the underlying space of X and apply Waldhausen's theory to obtain a spectrum $A(X)$; however, one would expect to be able to leverage the structure present in X to obtain a more refined invariant. My first result in Chapter 4 affirms that this is indeed the case.

Theorem 5.4. *If X is an orbispace, then $A(X)$ naturally refines to an orbispectrum $\underline{A}(X)$.*

This orbispectrum $\underline{A}(X)$ is closely related to (and indeed inspired by) the construction of the genuine equivariant algebraic K-theory of G-spaces [37, 38]. An ongoing research program of Malkiewich–Merling, together with Goodwillie and Igusa [25], seeks to establish an equivariant version of the stable parametrized h-cobordism theorem for G-manifolds. By work of myself, Chan, and Mejia [18], Malkiewich–Merling's construction also encodes equivariant invariants such as the Euler characteristic and Whitehead torsion, as studied by Lück, Illman, and many others [4, 9, 32, 35]. In Chapter 4, I show that $\underline{A}(X)$ is related to these G-equivariant theories, across all G, and the relevant G-equivariant invariants live in the G-fixed points $\underline{A}(X)^G$.

However, to obtain invariants that are truly sensitive to the full orbispace structure (rather than the purely G-equivariant structures), we need to pass to a suitable "limit" of $\underline{A}(X)^G$ over all G, which is a spectrum we denote by $A_{\mathcal{O}}(X)$. I show in Chapter 4 that $A_{\mathcal{O}}(X)$ indeed encodes the expected invariants. For instance, this K-theory extends the universal Euler characteristic beyond orbifolds to more general cell complexes, generalizing [26, Theorem 1].

Theorem 5.5. *For any orbispace X, the low-degree homotopy groups of $A_{\mathcal{O}}(X)$ encode versions of the Euler characteristic and Wall finiteness obstruction for orbispaces, exhibiting them as the universal additive invariants of orbispaces. For every finite group G, there is a map $A_{\mathcal{O}}(X) \to \underline{A}(X)^G$ which sends these invariants to their G-equivariant analogs.*

In light of the maps above, the spectrum $A_{\mathcal{O}}(X)$ can be thought of as piecing together the local G-equivariant information, compatibly across all G, to form genuine orbispace invariants. It is built explicitly as a K-theory spectrum associated to a category of *retractive orbispaces* over X. For example, when $X = *$, the spectrum $A_{\mathcal{O}}(*)$ is essentially the same as the K-theory of $\mathrm{orbiCW}_*^{\mathrm{fin}}$, and we can recover the universal Euler characteristic $\chi_{\mathcal{O}}$ from its connected components.

An important feature of $A_{\mathcal{O}}(X)$ is the following splitting result, which is reminiscent of a similar splitting in the equivariant setting [8].

Theorem 5.6. *There is an equivalence*

$$A_{\mathcal{O}}(X) \simeq \bigvee_{[G]} A(X^G_{h\,\mathrm{Aut}(G)}),$$

where the coproduct is over isomorphism classes of finite groups and $X^G_{h\,\mathrm{Aut}(G)}$ is a space obtained from the "G-equivariant part" of X.

This result is significant because it expresses the K-theory groups of an orbispace as a direct sum of K-groups indexed over all finite groups. Consequently, any orbifold invariants $f(X)$ encoded by $A_{\mathcal{O}}(X)$ may instead be understood via the corresponding *family of invariants* $\{f(X^G_{h\,\mathrm{Aut}(G)})\}$ which are, in theory, easier to compute.

I hope to leverage this work to prove a version of the s-cobordism theorem for orbifolds, which would provide algebraic obstructions to an h-cobordism of orbifolds being trivial. Proving an s-cobordism theorem for orbifolds is an ambitious goal, as even the purely equivariant setting introduces complications due to the lack of equivariant transversality [6, 35]. I outline some conjectures in this direction at the end of Chapter 4.

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