algebraic *K*-theory of orbispaces

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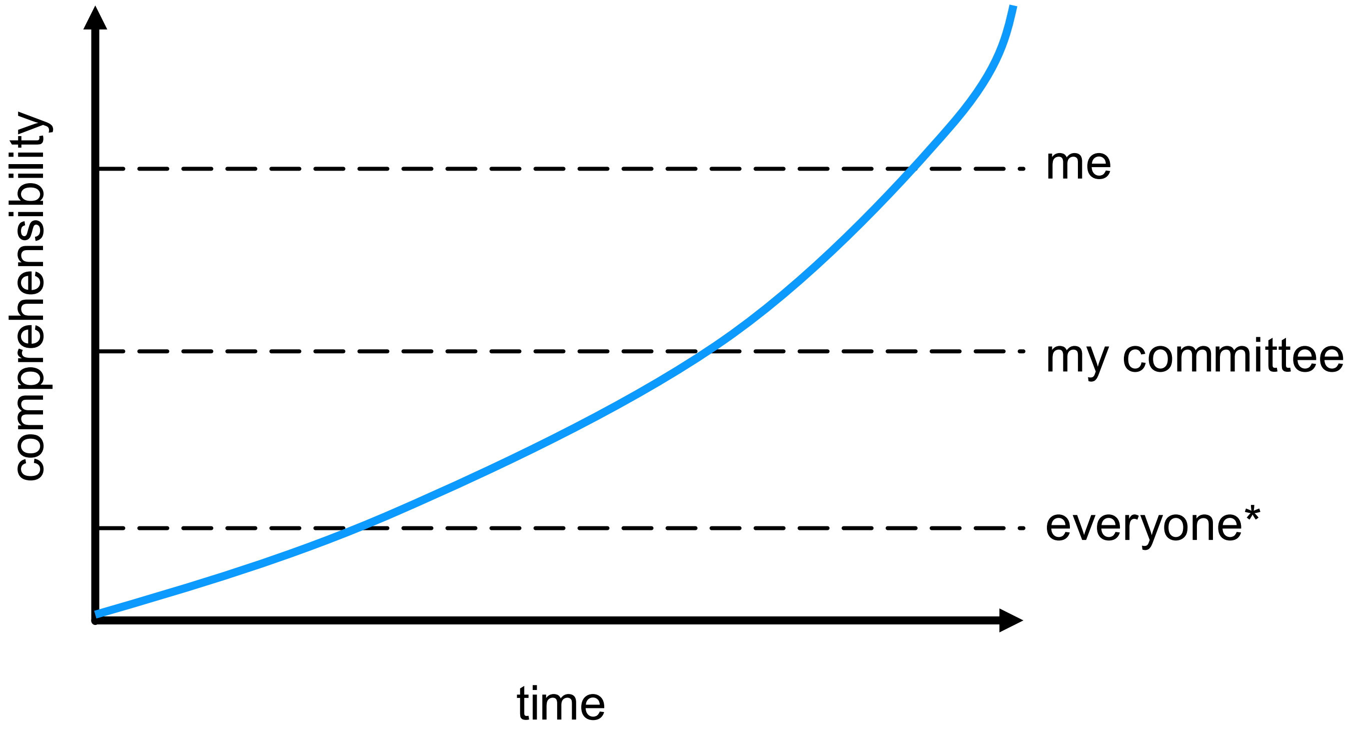
overarching goal

develop and study homotopy invariants of orbifolds that arise via Waldhausen's higher algebraic K-theory

talk outline

- Part I: Introduction
  - What are these three things?
  - What is known about them?
- Part II: Main results
  - What did I do with them?
  - What's next?

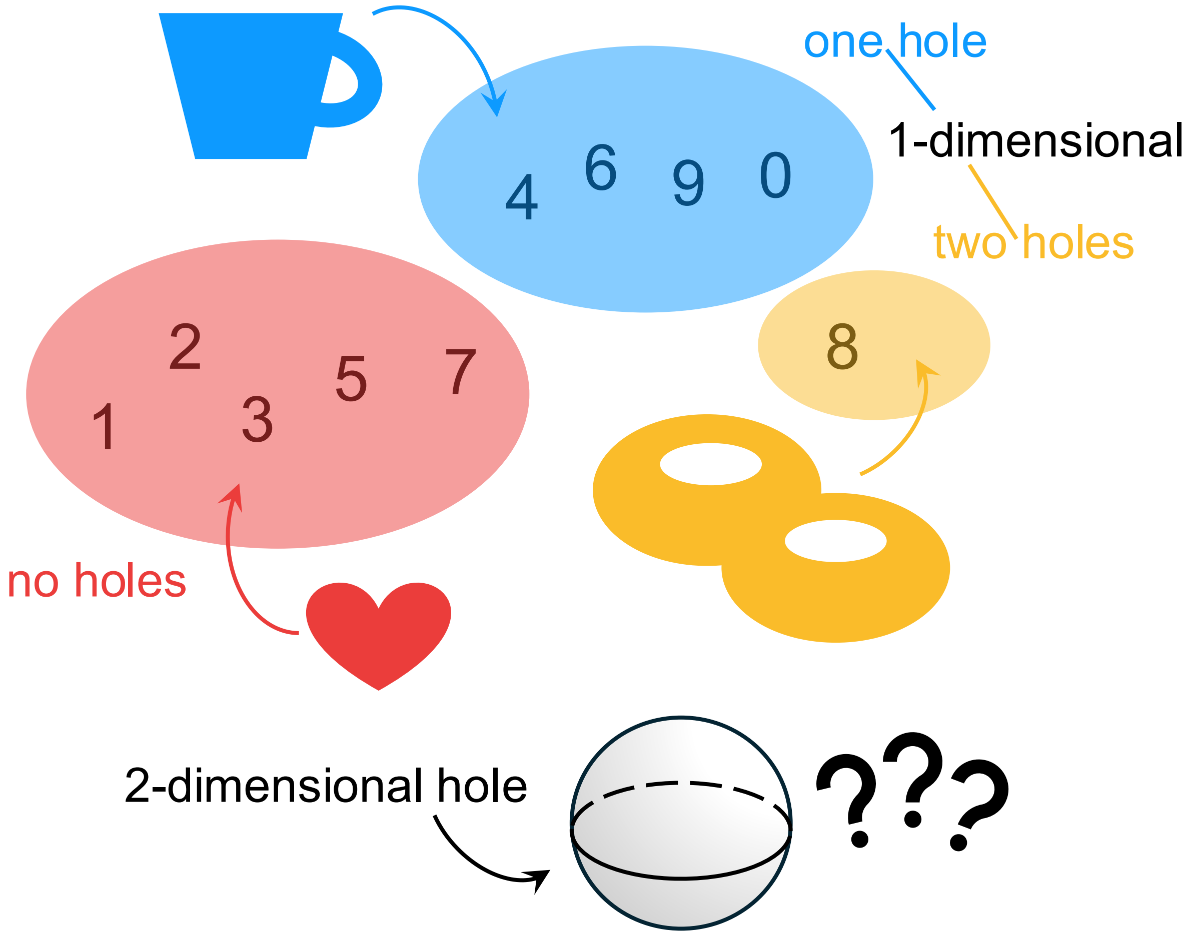


overarching goal

develop and study homotopy invariants of orbifolds that arise via Waldhausen's higher algebraic K-theory

idea

homotopy invariants detect the “essential structure” of a shape



overarching goal

develop and study **homotopy invariants** of orbifolds that arise via Waldhausen's higher algebraic *K*-theory

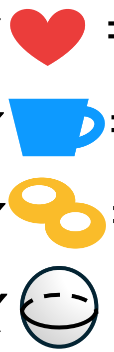
e.g. count "symmetric" holes

key example

the *Euler characteristic* is the alternating sum

$$\chi = b_0 - b_1 + b_2 - \cdots$$

where b_n is the number of "*n*-dimensional holes"



$\chi = 1 - 0 + 0 = 1$

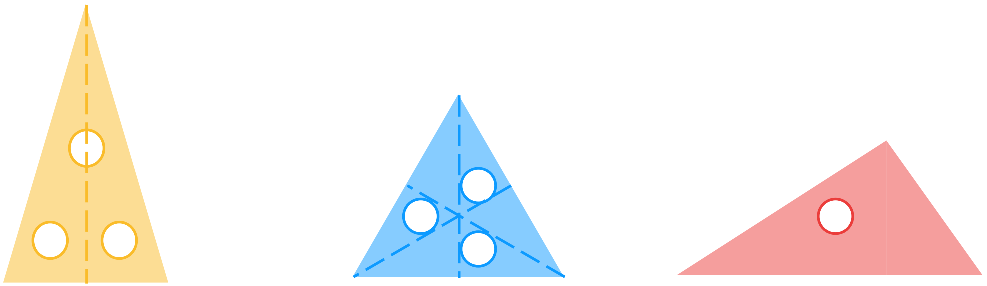
$\chi = 1 - 1 + 0 = 0$

$\chi = 1 - 2 + 0 = -1$

$\chi = 1 - 0 + 1 = 2$

motivating idea

develop refined "equivariant" invariants that see symmetries



reflectional symmetry

(C_2-symmetry)

reflectional + rotational symmetry

(D_6-symmetry)

no symmetry

(e-symmetry)

specifically: invariants that see "orbispace" structure

overarching goal

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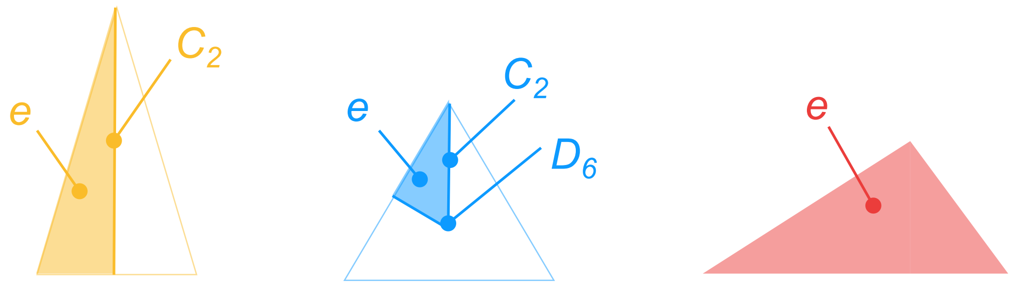
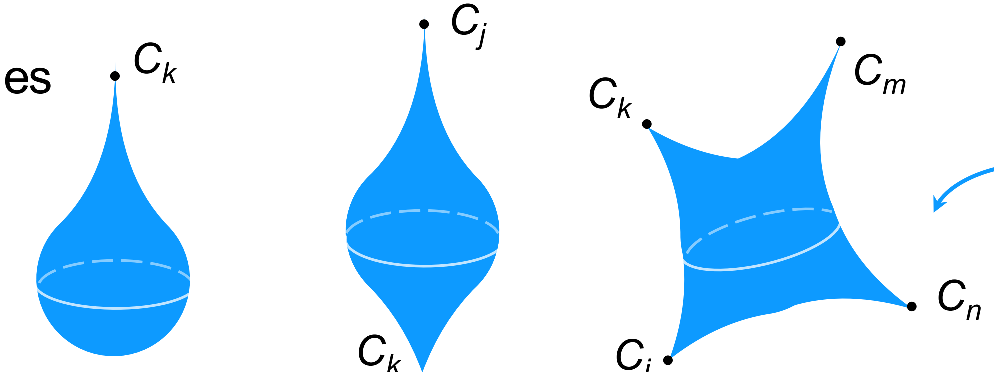
examples

spaces *X* with *G*-action give *global quotient* orbispaces *X//G*



examples



definition (Gusein-Zade, Luengo, Melle-Hèrnandez)

the *universal Euler characteristic* is

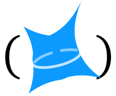
$$\chi(X) = \sum_{[G]} \chi(X^{(G)})[G] \in \mathcal{A}$$

??

(global Burnside ring)

$\chi($  $) = [e] + [C_k]$

$\chi($  $) = [C_k] + [C_j]$

$\chi($  $) = -2[e] + [C_k] + [C_j] + [C_n] + [C_m]$

$$(2 - \#\text{singularities})[e] + \sum [C_{k_i}]$$

overarching goal

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recall

if $f\colon \mathrm{CW}_*^{\mathrm{fin}} \to R$ is: (χ: $\mathrm{CW}_*^{\mathrm{fin}} \to \mathbb{Z}$, $\mathbb{Z} \to R$)

1. homotopy invariant
2. reduced $f(*) = 0$
3. additive $f(X) = f(A) + f(X/A)$

then f factors through the (reduced) Euler characteristic

observation (Waldhausen)

the Euler characteristic gives an isomorphism $K_0\left(\mathrm{CW}_*^{\mathrm{fin}}\right) \cong \mathbb{Z}$

definition (Gusein-Zade, Luengo, Melle-Hèrnandez)

the *universal Euler characteristic* is

$$\chi(X) = \sum_{[G]} \chi(X^{(G)})[G] \in \mathcal{A}$$

theorem (Gusein-Zade, Luengo, Melle-Hèrnandez)

the universal Euler characteristic has an analogous characterization

analogous observation

the (reduced) universal Euler characteristic gives an isomorphism $K_0\left(\mathrm{orbiCW}_*^{\mathrm{fin}}\right) \cong \mathcal{A}$

overarching goal

develop and study homotopy invariants of orbifolds that arise via Waldhausen's higher algebraic *K*-theory

observation (Waldhausen)

the Euler characteristic gives an isomorphism $K_0\left(\mathrm{CW}_*^{\mathrm{fin}}\right) \cong \mathbb{Z}$

Waldhausen's algebraic *K*-theory of spaces

every space *X* determines a spectrum *A(X)* that contains homotopical & geometric information

stable parametrized *h*-cobordism theorem (Waldhausen, Jahren, Rognes)

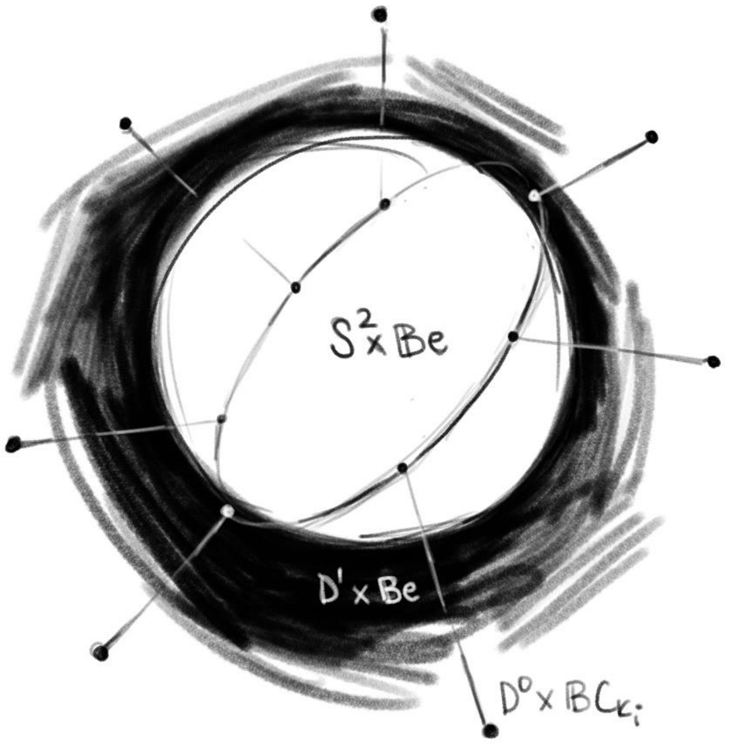
when *X* is a smooth manifold, *A(X)* is related to a space of *h*-cobordisms

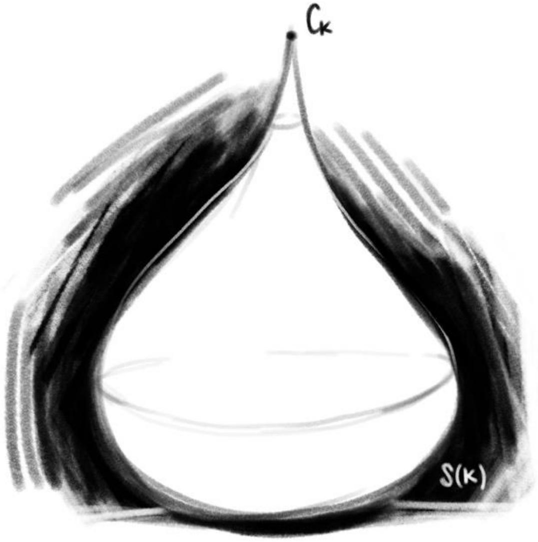
analogous observation

the (reduced) universal Euler characteristic gives an isomorphism $K_0\left(\mathrm{orbiCW}_*^{\mathrm{fin}}\right) \cong \mathcal{A}$

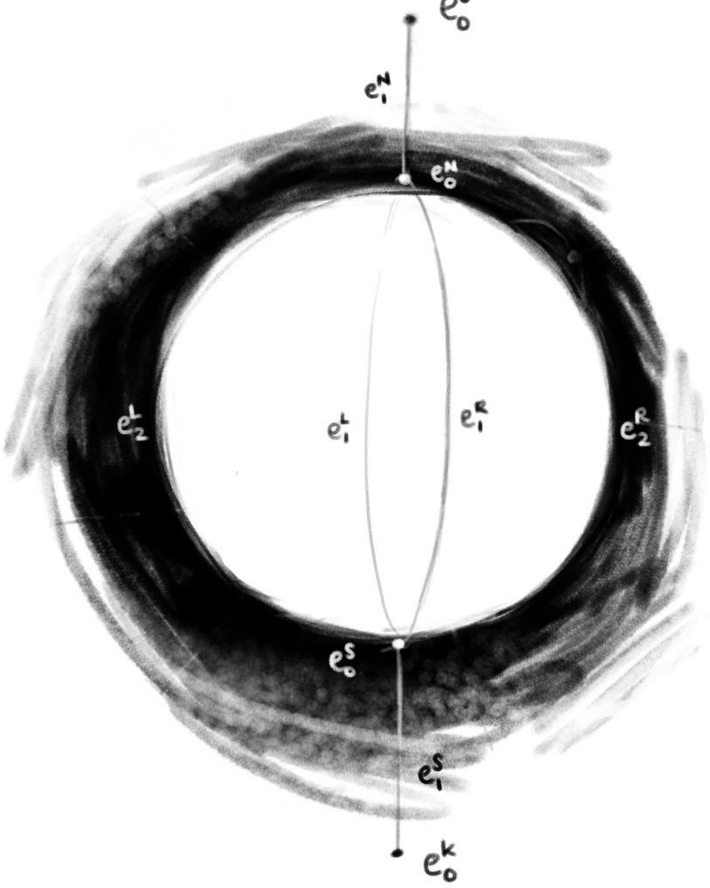
motivating questions for my work

1. if *X* is an orbispace, what structure does *A(X)* have?
2. what kind of geometric & homotopical invariants does it contain?
3. connection to orbifold *h*-cobordisms?





overview + summary of main results



motivating question #1

if *X* is an orbispace, what structure does *A(X)* have?

theorem

if *X* is an orbispace, then *A(X)* refines to an *orbispectrum*

theorem

orbispectra are “globally defined” spectral Mackey functors

corollary

the homotopy groups of an orbispectrum form a globally defined Mackey functor

(aka *biset* functor)

a table of analogies

non-equivariant	G-equivariant	orbivariant
space $X \in \mathcal{S}$	G-space $X \in \mathcal{S}_G$ $\mathcal{S}_G = Fun(O_G^{op}, \mathcal{S})$ (Elmendorf)	orbispace $X \in \mathcal{S}_O$ $\mathcal{S}_O = Fun(O^{op}, \mathcal{S})$ (Gepner--Henriques)
spectrum $E \in Sp$ infinite loop space $E \simeq \Omega^n X_n$	G-spectrum $E_G \in Sp_G$ G-infinite loop space $E_G \simeq \Omega^V X_V$ spectral G-Mackey functors $E_G \in Mack_G(Sp)$ (Guillou—May, Barwick)	orbispectrum $\underline{E} \in Sp_O$ G-infinite loop space for every G $res_G \underline{E} \simeq \Omega^V X_V$ (Schwede, Cnossen) spectral globally-defined Mackey functors $\underline{E} \in Mack_O(Sp)$
homotopy groups $\pi_k(E) \in Ab$	homotopy Mackey functors $\pi_k(E) \in Mack_G(Ab)$	homotopy biset functors $\pi_k(E) \in Mack_O(Ab)$

theorem

orbispectra are "globally defined" spectral Mackey functors

corollary

the homotopy groups of an orbispectrum form a globally defined Mackey functor

(aka *biset* functor)

theorem

$\mathrm{Mack}_{\mathrm{orb}}(\mathrm{Ab}) \simeq$ Bouc/Webb's globally-defined Mackey functors

definition (Bouc, Webb)

a globally-defined Mackey functor $\underline{A}$ is specified by:

- an Abelian group $\underline{A}(G)$ for each finite group G (e.g. representation ring, group cohomology, …)
- for each $H \hookrightarrow G$, restrictions and transfers

$$\underline{A}(H) \underset{R}{\overset{T}{\rightleftarrows}} \underline{A}(G)$$

definition

let $\mathcal{C}$ be an (infinity) category with products and define

$$\mathrm{Mack}_O(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{F})^{op}, \mathcal{C})$$

coproducts-to-products

finite groupoids + faithful functors

idea

$BG \mapsto \underline{A}(G)$

$(BG \leftarrow BH = BH) \mapsto T$

$(BH = BH \rightarrow BG) \mapsto R$

remark: $\mathrm{Span}(\mathcal{F})$ is like the "un-group completed" version of Bouc/Webb's biset category

theorem

orbispectra are “globally defined” spectral Mackey functors

corollary

the homotopy groups of an orbispectrum form a globally defined Mackey functor

(aka *biset* functor)

theorem

$\mathrm{Mack}_{\mathrm{orb}}(\mathrm{Ab}) \simeq$ Bouc/Webb’s globally-defined Mackey functors

definition (Bouc, Webb)

a ...

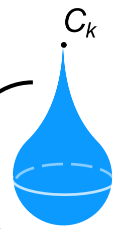
sp...

side quest

every globally-defined Mackey functor $\underline{A}$ gives an *Eilenberg—Maclane orbispectrum* $\underline{HA}$

theorem

these orbispectra represent ordinary cohomology theories for orbispaces



$$\tilde{H}_*\left(\bullet , \underline{A} \right) \cong \begin{cases} \underline{A}(C_k) & *= 0, \\ 0 & *= 1, * \geq 3, \\ \underline{A}(e) & *= 2 \end{cases}$$

$(BH = BH \to BG) \mapsto R$

“group completed” version of Bouc/Webb’s biset category

theorem

orbispectra are "globally defined" spectral Mackey functors

theorem

every connective orbispectrum is the algebraic *K*-theory of some "globally defined" categorical Mackey functor

definition/theorem

if *X* is an orbispace, then *A(X)* refines to an *orbispectrum* *A*(*X*)

$$\underline{A}(X) \leftrightsquigarrow \{A_G(\mathrm{res}_G(X))\}_G$$

G-equivariant K-theory of Malkiewich--Merling

definition

let $\mathcal{C}$ be an (infinity) category with products and define

$$\mathrm{Mack}_O(\mathcal{C}) \coloneqq \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{F})^{op}, \mathcal{C})$$

coproducts-to-products

finite groupoids + faithful functors

idea (Bohmann--Osorno, Barwick, Malkiewich--Merling, …)

let $K: \mathcal{C} \to \mathrm{Sp}$ be an algebraic *K*-theory functor ($\mathcal{C} = \mathrm{Sym}, \mathrm{Wald}, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}, \ldots$) and define

$$K_O\colon \mathrm{Mack}_O(\mathcal{C}) \to \mathrm{Mack}_O(\mathrm{Sp})$$

examples

- $\mathbb{S}_O \simeq K_O$(finite groupoids)
- suspension orbispectrum of $X \simeq K_O$(finite groupoids over *X*)
- <u>HA</u> $\simeq K_O$(<u>A</u>)

motivating question #1

if *X* is an orbispace, what structure does *A(X)* have?

definition/theorem

if *X* is an orbispace, then *A(X)* refines to an *orbispectrum* $\underline{A}(X)$

$$\underline{A}(X) \rightsquigarrow \{A_G(\mathrm{res}_G(X))\}_G$$

motivating question #2

what kind of geometric & homotopical invariants does $A_{orb}(X)$ contain?

corollary (Lück, C.—Chan—Mejia)

the *G*-fixed points of $\underline{A}(X)$ house *G*-equivariant invariants

example: X=∗

$$\chi_G \in \pi_0 \underline{A}(*)^G \cong K_0\left(G\mathrm{CW}_*^{\mathrm{fin}}\right)$$

$$\pi_0 A_{orb}(X) \cong$$

$$\chi_{\mathrm{orb}} \in K_0\left(\mathrm{orbi-CW}_*^{\mathrm{fin}}\right) \cong \mathcal{A}$$

$$\chi \in \pi_0 \underline{A}(*)^e \cong K_0\left(\mathrm{CW}_*^{\mathrm{fin}}\right)$$

theorem

there is a complementary spectrum $A_{orb}(X)$ which contains genuine orbispace invariants, and a map

$$A_{orb}(X) \to \lim_G \underline{A}(X)^G$$

motivating question #2

what kind of geometric & homotopical invariants does $A_{orb}(X)$ contain?

answer

versions of the Euler characteristic and Wall's finiteness obstruction which can be understood via a family of non-equivariant invariants

conjecture

$\pi_1 A_{orb}(X)$ contains a version of Whitehead torsion that sees "simple" equivalences

motivating question #3

what connection do these constructions have to orbifold *h*-cobordisms?

theorem

there is a complementary spectrum $A_{orb}(X)$ which contains genuine orbispace invariants, and a map

$$A_{orb}(X) \to \lim_{G} \underline{A}(X)^{G}$$

theorem (using jt. work with Chan—Chedalavada—Mejia)

there is a splitting $A_{orb}(X) \simeq \bigvee_{[G]} A(X^{G}_{h\Gamma(G)})$

built from "G-part" of *X*

example: $X = *$

$$A_{orb}(*) \simeq \bigvee_{[G]} A(B\mathrm{Out}(G) \times B^2 Z(G))$$

conjecture

$\pi_1 A_{orb}(X)$ contains a version of Whitehead torsion that sees “simple” equivalences

motivating question #3

what connection do these constructions have to orbifold *h*-cobordisms?

conjecture

under suitable dimension hypotheses, Whitehead torsion classifies orbifold *h*-cobordisms

conjecture

$A_{orb}(X)$ fits into a stable parametrized orbifold *h*-cobordism theorem

theorem

there is a complementary spectrum $A_{orb}(X)$ which contains genuine orbispace invariants, and a map

$$A_{orb}(X) \to \lim_{G} \underline{A}(X)^G$$

theorem (using jt. work with Chan—Chedalavada—Mejia)

there is a splitting $A_{orb}(X) \simeq \bigvee_{[G]} A(X^G_{h\Gamma(G)})$

stable parametrized *h*-cobordism theorem

if M is a manifold, then there is a (split) fiber sequence

$$\Sigma^\infty_+ M \to A(M) \to \mathrm{Wh}(M)$$

where $\Omega^{\infty+1}\mathrm{Wh}(M) \simeq \mathcal{H}^\infty(M)$ is a space of stable *h*-cobordisms

thank you all